Optimal Robust Divisible Mechanisms for Public Goods*

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Abstract

Consider a mechanism for robust allocation of a divisible public good among n agents, when the budget is exactly balanced. In the linear[1] framework, we show that such mechanisms are equivalent to lotteries over simple[2] posted prices, as long as the allocation function is upper semi-continuous. We further show that, for any prior beliefs, the maximum of expected welfare is attained at the extreme point of this family. In other words, simple posted prices are robustly optimal.

Two existing results are relevant to ours. The first one is the lottery representation of robust mechanisms by Hagerty and Rogerson (1987) when the allocation is either binary or differentiable. This classic result was obtained in the context of bilateral trade, which can be thought of as a public good for two agents.

The second one is the robust optimality of threshold[3] mechanisms by Kuzmics and Steg (2017), obtained in the public goods setting, but only for binary allocations. We generalize both of these results, since binary and differentiable functions are special cases of upper semi-continuous ones.

Keywords: optimal, robust, divisible, public goods, budget balance.

1 Introduction

We study *divisible* allocation of a *pure* (non-excludable) public good among n agents with quasi-linear utility. Divisibility means that the good can be allocated at different levels, in contrast with the *binary* allocation that can pick either full scale or nothing.

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[1]quasi-linear utilities, constant marginal utilities and constant marginal costs

[2]with a binary and upper semi-continuous allocation function

[3]threshold mechanisms are equivalent to simple posted prices if the allocation function is upper semi-continuous

Non-excludability suggests that everyone has an unrestricted access to the full amount produced.

Such public goods are easy to find in any modern community, the simplest example being public security, which can vary in quality and, therefore, costs of provision. We restrict attention to the special case when both the costs and the utilities are linear in production level.

We are interested in finding an optimal robust mechanism, meaning that expected welfare is maximized subject to incentive compatibility (IC), individual rationality (IR), and budget balance (BB) constraints that have to be satisfied for all realization of types, or, in other words, *ex-post*. The budget balance property indicates that the mechanism never produces deficit or surplus. This definition of robustness is in line with Hagerty and Rogerson (1987) and Copic and Ponsati (2016), but there are several other interpretations such as in Bergemann and Morris (2005), Bergemann and Schlag (2011), or Borgers and Smith (2012). Instead of the ex-post IC, a dominant-strategy incentive compatibility ($DSIC$) constraint is commonly used in the literature. These two types of constraints are equivalent for direct mechanisms in private values environments like ours.

It was recently shown in Kuzmics and Steg (2017) that robust mechanisms for public good, which minimize expected welfare loss among binary ones, have to take a certain threshold form. However, it is not obvious that such mechanisms should be binary in the first place. The goal of our paper is to study mechanisms in the public goods setting, that are robustly optimal among the divisible ones.

Our contribution consists of two main results. The first main result is that, under certain additional assumptions, expected welfare loss attains minimum at a simple posted price, or, equivalently, a threshold mechanism. This finding reinforces the results in Kuzmics and Steg (2017) by justifying the restriction to binary mechanisms in two steps. First, in Proposition 1, we show that a robust divisible mechanism is a lottery over simple posted prices. Second, in Proposition 2, we argue that, for any prior belief, expected welfare loss is maximized at the vertex of the lottery simplex. Our second main result is that, under certain beliefs, a significant portion of welfare is guaranteed to be lost even if the mechanism was chosen optimally among the robust ones. In Proposition 3, we derive a lower bound for the associated welfare loss, which grows linearly in the number of agents. This bound is tight for the case of two agents.

Most of the assumptions required for our results are of technical nature and barely restrain their applicability. But there is one substantial restriction, namely Assumption 3, which requires that each agent's equilibrium utility at the lowest type is equal to zero. A similar property was derived in Kuzmics and Steg (2017) as a result of expected welfare loss minimization, for binary mechanisms. Due to complexity of the divisible good setting, we do not derive it, instead in Section 3.1, we argue that in a variety of situations it is automatically satisfied.

On the technical side, our approach is closest in spirit to Hagerty and Rogerson (1987), who studied robust mechanisms for bilateral trade. This problem is dual to that of a public good with two agents, and so our results can be considered as a direct extension.

A distinct feature of our paper is that we consider a broad class of upper semi-continuous allocation functions, as opposed to differentiable ones in Hagerty and Rogerson (1987), or binary ones in Kuzmics and Steg (2017). These two classes of mechanisms typically involve different solution techniques. In contrast, we develop a unified method that suits both, which is demonstrated in the proof of Lemma 2.

The tools are novel to the mechanism design literature, and can be possibly applied to other economic environments.

1.1 Literature review

There are multiple opinions in the literature on what robustness exactly means. For example, in Bergemann and Morris (2005), robustness is defined as Bayesian implementation on all type spaces, which is a stronger concept than ex-post implementation. An alternative method was proposed in Borgers and Smith (2012), which eliminated weakly dominated strategies based only on the knowledge of the support of distribution. Robustness in Bergemann and Schlag (2011) means that the maximal regret from a small misspecification of beliefs is also relatively small.

Robustness in our paper implies that the incentive compatibility, individual rationality, and budget balance constraints are formulated in such a way that they do not rely on common knowledge of the distribution of types. In the context of direct mechanisms, this amounts to all the constraints being evaluated ex-post. This is the exactly same approach that was used in Hagerty and Rogerson (1987) to study robust bilateral trade.

In the classic papers by Myerson and Satterthwaite (1983) and Guth and Hellwig (1986), the optimal Bayesian (second-best) divisible mechanisms were identified for bilateral trade and pure public goods respectively. The optimal robust (third-best) divisible mechanisms were described in Hagerty and Rogerson (1987) and further studied in Copic and Ponsati (2016). A systematic analysis of the second-best and third-best mechanisms for bilateral trade, which can also be thought of as a pure public good with two agents, can be found in Borgers et al. (2015).

Related are numerous papers that aim at selecting a thin set of mechanisms without any particular welfare assessment. Typically, a strong solution concept like coalition-proof stability as in Moulin (1994) and Bierbrauer and Hellwig (2016), or a symmetry/anonymity assumption as in Serizawa (1999) is used. Compared to these papers, we harness a relatively weak solution concept, that allows to select a rich but tractable family of asymmetric mechanisms.

The optimality of posted prices in our paper is similar to the result in Copic and Ponsati (2016), but there are a few substantial differences. On the one hand, they consider a broader class of welfare criteria captured by the notion of ex-post Pareto optimality. On the other, they insist on the interpretation of mechanisms as lotteries, which allows them to use a stronger version of ex-post IC and ex-post IR constraints. Our interpretation of mechanisms is that the allocation is divisible, and so it is more general and consistent with the classic approach in Hagerty and Rogerson (1987).

Our assessment of welfare losses of robust mechanisms is similar to how revenue losses of certain simple mechanisms were studied by Hart and Nisan (2017) in the context of monopolistic screening. Instead of directly solving for the profit, they show that a constant fraction of optimal (second-best) revenue can be attained, for all prior beliefs. Likewise, in our setting, a constant fraction of optimal (second-best) welfare is forgone by robust mechanisms, under certain beliefs.

The rest of the paper is organized as follows. We describe the setup in Section 2, and the preliminary results in Section 3. The latter consists of three parts, in which we discuss the additional assumptions, introduce the tools for studying divisible mechanisms, and prove the technical lemmas needed for the main results of the paper. The main results are in Section 4, which consists of three parts: the characterization of robust mechanisms in terms of lotteries over posted prices in Section 4.1, the proof that posted prices are optimal in Section 4.2, and the derivation of the welfare bounds in Section 4.3. We discuss our findings in Section 5.

2 Setup

A divisible public good can be produced at any level in the segment $[0,1]$, where zero stands for no production, and the marginal cost of production is equal to $c > 0$.

Let there be n agents with quasi-linear utilities. Each agent has constant marginal valuation from the public good, which is unknown to the designer. Denote by $v = \{v_i\}$ the profile of valuations, and let it be distributed on a rectangular area $V = \{v : v_i \in [a_i, b_i]\}$, where $\sum a_i < c < \sum b_i$.

Following the tradition of mechanism design, we focus on a direct mechanism that assigns the level of production and transfers based on the reported profile of types v. We consider a divisible (as opposed to binary) mechanism, which means that for a profile of types v a certain level of production between 0 and 1 is assigned.

Definition 1 *A* ***direct mechanism*** μ *is a mapping from* V *into* $[0,1] \times \mathbb{R}^n$*:*

$$\mu : v \mapsto (\varphi, \tau)$$

A mechanism μ maps the profile of valuations v into $\varphi(v)$ - the expected allocation function, and $\tau = \{\tau_i\}_{i=1}^n$ - the vector of expected payments made by the agents. For the rest of the paper, two mechanisms are considered equivalent if their allocation function and transfers coincide.

For the reported profile $v = (v_i, v_{-i})$ and a true type $\hat{v}_i$ of agent i, his off-equilibrium utility $U_i(\hat{v}_i, v)$ is:

$$U_i(\hat{v}_i, v) = \hat{v}_i \varphi(v) - \tau_i(v)$$

For the truthfully reported profile of types $v = (v_i, v_{-i})$ the equilibrium utility $U_i(v)$ and the equilibrium budget $B(v)$ are:

$$U_i(v) = U_i(v_i, v), \quad B(v) = \sum_i \tau_i(v) - \varphi(v)c$$

Definition 2 *A mechanism μ is **robust** (with budget balance) if it satisfies:*

1. *ex post IC: $\hat{v}_i \in arg\max_{v_i} U_i(\hat{v}_i, v)$,*
2. *ex post IR: $U_i(v_i, v) \geqslant 0$,*
3. *ex post BB: $B(v) = 0$,*

for all v and $\hat{v}_i$ in the support.

A standard exercise in mechanism design is to characterize an ex-post *IC* mechanism as a non-decreasing allocation function $\varphi(v)$ such that utilities and transfers satisfy:

$$U_i(v_i'', v_{-i}) - U_i(v_i', v_{-i}) = \int_{v_i'}^{v_i''} \varphi(z, v_{-i}) dz \tag{1}$$

$$\tau_i(v_i'', v_{-i}) - \tau_i(v_i', v_{-i}) = \int_{v_i'}^{v_i''} z d\varphi(z, v_{-i}) \tag{2}$$

for all v_i', v_i'', v_{-i} in the support.

Formulas (1) and (2) represent the same enveloping conditions in the quasi-linear setting, see Hagerty and Rogerson (1987) or Myerson and Satterthwaite (1983) for the discussion. A mechanism is additionally ex-post *IR*, if and only if for every agent i:

$$U_i(a_i, v_{-i}) \geqslant 0 \tag{3}$$

because the equilibrium utility is non-decreasing in own type. Finally, the mechanism is ex post *BB* if the total surplus is equal to the sum of equilibrium utilities:

$$B(v) = (\sum v_i - c)\varphi(v) - \sum U_i(v_i, v) = 0 \tag{4}$$

Conditions (1)-(4) will be the starting point of our analysis in the next section.

3 Preliminary results

Our first observation about the family of robust mechanisms is that they form a convex cone (with the origin at the autarky) in the space of mechanisms (φ, τ). That is because a convex combination of any collection of ex-post *IC*, *IR* and *BB* mechanisms also satisfies these properties, which follows from formulas (1)-(4).

For analytical convenience, we will often consider *randomized* mechanisms, that are distributions over a certain family of mechanisms. The hierarchy of robust mechanisms in our paper is therefore:

$$\textit{binary} \subset \textit{randomized binary} \subset \textit{divisible} = \textit{randomized divisible}$$

The fact that robust divisible mechanisms form a convex set, however, is of little use unless we know the minimal set of mechanisms that span the whole family through convexification, which will be the goal of Proposition 1.

To find this minimal set we need several additional assumptions that are described in Section 3.1, and a few technical tools and lemmas listed in Section 3.2 and Section 3.3.

3.1 Additional assumptions

Here we discuss in detail the four assumptions used in the paper.

Assumption 1 *A mechanism is* ***non-wasteful*** *if* $sup_v \varphi(v) = 1$.

This assumption was used in Copic and Ponsati (2016) and is merely a normalization of the mechanism. Clearly, any robust mechanism such that $sup_V \varphi(v) \in (0, 1)$ can be appropriately scaled to be non-wasteful, which would also be an improvement from the social point of view.

Assumption 2 *The allocation function* $\varphi(v)$ *is* ***upper-semicontinuous*** *on* V *if:*

$$\limsup_{v \to v_0} \varphi(v) = \varphi(v_0)$$

This assumption was used in Dobzinski et al. (2017) in the context of binary mechanisms and is yet another normalization. For non-decreasing functions it is equivalent to *right-continuity* in each coordinate v_i separately, or in the vector of coordinates v as a whole. The role of this assumptions is also discussed in Section 4 in the definition of a simple posted price.

Assumption 3 *For every* i *there exists a* ***pivotal type*** $\xi(v_{-i})$ *such that* $U_i(\xi, v_{-i}) = 0$.

This assumption requires that for any realization of types v_{-i} there exist a type of agent i that is just indifferent to participate, and, crucially, this pivotal type should not depend on the actual type of agent i. Though seemingly artificial, this is a fairly natural property of the mechanism. In the context of binary mechanism, it was derived from a variety of different assumptions in Kuzmics and Steg (2017). For divisible mechanisms it can be derived from one of the following assumptions on the primitives of the model:

3.1 *Largest type space*: $V = \mathbb{R}^n$,

3.2 *Veto power*: $a_i + \sum_{j \neq i} b_j < c$, for all i,

3.3 *Cost sharing*: $a_i = 0$ and $\tau_i(v) \geqslant 0$, for all i.

If the public good can be disliked by an agent, then it is natural to pick the largest possible type space $V = \mathbb{R}^n$. We can think of Assumption 3.1 as part of the robustness concept, that requires the mechanism to work for distributions with unknown supports. It then follows that the pivotal type of agent i is $\xi(v_{-i}) = c - \sum_{j \neq i} v_j$.

A slightly weaker version of Assumption 3.1 is Assumption 3.2, that does not require negative valuations. It posits that, for any agent, there exists a type that will make the public good inefficient no matter what the types of other agents are. That means that every agent has a power to veto the public good by announcing just below his pivotal type $\xi(v_{-i}) = c - \sum_{j \neq i} v_j$, which is closely related to the *instance of full veto power* assumption in Kuzmics and Steg (2017).

Finally, if the setting implies that the valuations are nonnegative, that is $a_i = 0$, the restriction to non-positive transfers as in Assumption 3.3 guarantees that a situation where somebody who does not care about the public good earns free money by just stepping into the mechanism would not occur. The pivotal type in this situation is $\xi_i(v_{-i}) = \max\{c - \sum_{j \neq i} v_j, a_i\}$.

Assumption 4 *$a_i = 0$, $b_i = b < c$, for all i.*

This assumption tells that the support is an equilateral cube with bounds such that nobody can have marginal value above marginal costs, and the lowest marginal value exactly zero. We use this assumption is far from necessary for our results, but we invoke it for the welfare loss bounds in Proposition 2 to have a concise analytical form.

3.2 Technical tools

In order to formulate and derive the main results of the paper we need to introduce some technical tools, such as *n-th mixed difference operator* and a *distribution function*, adapted from Chapter 3 in the Shiryaev (2016) textbook. These are necessary for the construction of a random variable out of its conjectured *CDF*, which will be at the core of Proposition 1. In the context of binary mechanisms such as studied in Kuzmics and Steg (2017), these tools are redundant since no randomization takes place.

Consider an n-dimensional (not necessarily equilateral) cube $C \subset V$ with left-open and right-closed boundaries:

$$C = \{v \in V \mid v_i \in (\tilde{a}_i, \tilde{b}_i], \ \forall i\}$$

and let $\mathcal{C}$ be the set of all such cubes C in V. The following definition generalizes the concept of a mixed n-th derivative to functions that do not have to be differentiable.

Definition 3 *For a cube $C \in \mathcal{C}$ the n-th mixed difference operator $D_C^{1,\dots,n}$ is defined as:*

$$D_C^{1,\dots,n}\varphi = D^1_{(\tilde{a}_1,\tilde{b}_1]}(\dots(D^n_{(\tilde{a}_n,\tilde{b}_n]}\varphi(v)))$$
$$D^k_{(\tilde{a}_k,\tilde{b}_k]}\varphi(v) = \varphi(\tilde{b}_k, v_{-k}) - \varphi(\tilde{a}_k, v_{-k})$$

Note that each separate coordinate-wise difference is an operator acting on the space of functions on V, but the n-th mixed difference maps an allocation function into a constant and thus is a functional. If $\varphi(v)$ is a *CDF* of a random variable, then $D_C^{1,\dots,n}\varphi$ measures the probability of that random variable landing inside the cube C.

One useful property of $D_C^{1,\dots,n}$ is that for two adjacent (sharing the same face) cubes C_1 and C_2 the sum of the two corresponding functionals is equal to the functional that of the merged cube $C_1 \cup C_2$:

$$D_{C_1 \cup C_2}^{1,\dots,n}\varphi = D_{C_1}^{1,\dots,n}\varphi + D_{C_2}^{1,\dots,n}\varphi. \quad (5)$$

In the case of random variables this means that the probability of landing in either of the two cubes is the same as the sum probabilities of landing in each cube separately.

The following definition captures the idea of the cumulative distribution function of a random variable, adapted to a special case when the domain is a bounded subset of the Euclidean space such as V.

Definition 4 *A function φ on V is a distribution function if and only if:*

1. $\inf_{v \in V} \varphi(v) = 0$, $\sup_{v \in V} \varphi(v) = 1$,
2. *it is right-continuous in the profile of types v,*
3. *the n-th mixed difference $D_C^{1,\dots,n}$ is non-negative for any $C \in \mathcal{C}$.*

Note that for the special case of a non-decreasing allocation function $\varphi(v)$, the second property is equivalent to upper-semicontinuity. The third property is equivalent to non-wastefulness, because there is always a region in V where production is inefficient and therefore $\varphi(v) = 0$.

3.3 Technical lemmas

Let E be the part of the type space, where production of public good is efficient or break-even, and its complement $\bar{E}$ - where production is strictly inefficient:

$$E = \{v \in V \mid \sum_i v_i \geqslant c\}, \quad \bar{E} = \{v \in V \mid \sum_i v_i < c\}$$

The following lemma takes advantage of Assumption 3 and obtains a concise analytical transcription of the budget balance property, as an integral equation with the allocation function as an unknown.

Lemma 1 *Under Assumption 3, if a mechanism is ex post IR, IC, BB then:*

$$(\sum_i^n v_i - c)\varphi(v) - \sum_{i=1}^n \int_{\xi_i(v_{-i})}^{v_i} \varphi(z, v_{-i}) dz = 0, \tag{6}$$

where $\xi_i(v_{-i})$ is the pivotal type, and $v \in E$.

Proof. Consider the budget surplus as in formula (4). By formula (1), agent i's ex post utility can be represented as an integral over the allocation function $\varphi(\cdot, v_{-i})$ with the bounds of integration $\xi_i(v_{-i})$ and v_i plus the utility at the boundary, that must be equal to zero. ∎

The following lemma further transforms the budget balance condition into a property of the mixed difference operator. This property will be the bedrock for the construction of the random variable in the main section.

Lemma 2 *Under Assumption 2, 3 if a mechanism is ex post IR, IC, BB then the n-th mixed difference* $D_C^{1,\dots,n}\varphi$ *is nonnegative for any cube* $C \in \mathcal{C}$*, and zero if additionally the cube fully belongs to either* E *or* $\bar{E}$*.*

Proof. Note first that $D_C^{1,\dots,n}\varphi = 0$ for any cube $C \subset \bar{E}$, because production in that region is inefficient and therefore $\varphi(v) = 0$.

Let do the same for $C \subset E$. If the allocation function is smooth in E, differentiate both sides of equation (6) with respect to every variable:

$$\sum_i \left(\frac{\partial^{n-1}}{\partial v_1 \dots \partial \hat{v}_i \dots \partial v_n} \frac{\partial}{\partial v_i}(v_i \varphi(v_i, v_{-i})) - c \frac{\partial^n}{\partial v_1 \dots \partial v_n} \varphi(v) \right) -$$
$$- \sum_i \frac{\partial^{n-1}}{\partial v_1 \dots \partial \hat{v}_i \dots \partial v_n} \left(\frac{\partial}{\partial v_i} \int_{c - \sum_{j \neq i} v_j}^{v_i} \varphi(\xi, v_{-i}) d\xi \right) = 0$$

after cancelling out the terms sides and dropping $(\sum_i v_i - c)$ we get:

$$\frac{\partial^n}{\partial v_1 \dots \partial v_n} \varphi(v) = \ 0 \tag{7}$$

that should hold for all $v \in E$. Integrating over the area of the cube $C \subset E$ we then obtain the necessary equation. See Appendix for the general proof with upper-semicontinuous functions.

Finally, let's show that $D_C^{1,\dots,n}\varphi \geqslant 0$ for any cube $C \subset V$. Using formula (5) and the previous result we can replace a cube C that crosses the hyperplane $\sum v_i = c$ with another cube C' (see Figure 1 for a graphical example) such that:

$$D_C^{1,\dots,n} = D_{C'}^{1,\dots,n}, \quad C' = \{v \in V \mid v_i \in (\tilde{a}_i, \eta_i],\ \forall i\}, \quad \eta_i = \min\{c - \sum_{j \neq i} \tilde{a}_j, \tilde{b}_i\}.$$

For the new cube we can establish the necessary property:

$$D_{C'}^{1,\dots,n} = \varphi(\eta) - \sum_i \varphi(c - \sum_{j \neq i} \eta_j, \eta_{-i}) \cdot \mathbb{I}(\tilde{b}_i \geqslant c - \sum_{j \neq i} \tilde{a}_i) \geqslant B(v) = 0$$

which follows from formula (6) and monotonicity of $\varphi(v)$. ■

4 Main results

To state the main results of the paper, we need to formally define a special robust mechanism that we refer to as a *simple posted price* throughout the paper. Fix a vector of prices $\rho = \{\rho_i\}_{i=1}^n$ such that $\sum \rho_i \geqslant c$.

Definition 5 *A mechanism* $\mu^{sp}(\rho) = (\varphi^{sp}(\rho), \tau^{sp}(\rho))$ *is a* ***simple posted price*** *if:*

$$\varphi^{sp}(\rho, v) = \prod_i \mathbb{I}(v_i \geqslant \rho_i), \quad \tau^{sp}(\rho, v) = \rho \cdot \varphi^{sp}(\rho, v)$$

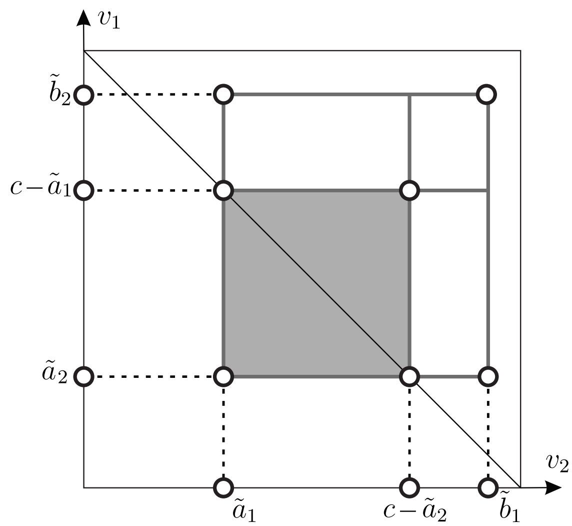

Figure 1: Cubes $C = [\tilde{a}_1, \tilde{b}_1) \times [\tilde{a}_2, \tilde{b}_2)$ and $C' = [\tilde{a}_1, c - \tilde{a}_2) \times [\tilde{a}_2, c - \tilde{a}_1)$ for the case $n = 2, V = [0,1]^2$ and $c = 1$.

For any given ρ, the allocation function $\varphi^{sp}(\rho, v)$ is simply an indicator function with a rectangular support. If the support were strictly above the hyperplane $\sum_i v_i = c$ as in Figure 2 (left), then the mechanism would generate budget surplus, and if it only touches the hyperplane as in Figure 2 (right), then it is exactly budget balanced.

Our definition of the *simple posted price* is in line with the definition of a *posted price* in Copic and Ponsati (2016), but we use the word *simple* to stress that there are many more robust mechanisms that can be referred to as *posted prices.* In the context of binary outcomes, this broader class of mechanisms is referred to as *threshold mechanisms* in Kuzmics and Steg (2017).

4.1 Characterization

Proposition 1 *There is a one-to-one correspondence between the family of robust divisible mechanisms satisfying Assumptions 1,2,3 and distributions over simple posted prices. The allocation function of such mechanism coincides with the respective cumulative distribution function.*

Proof. It's fairly easy to see that the allocation function of a single posted price μ_p is the *CDF* of a corresponding lottery with a single outcome ρ. Same property holds for any lottery or distribution over simple posted prices:

$$Prob(\rho : \rho_i \leqslant v_i \text{ for all } i) = \varphi(v).$$

To show the converse we will adapt a theorem in Shiryaev (2016) that states that there is a one-to-one correspondence between measures on V and the distribution functions as

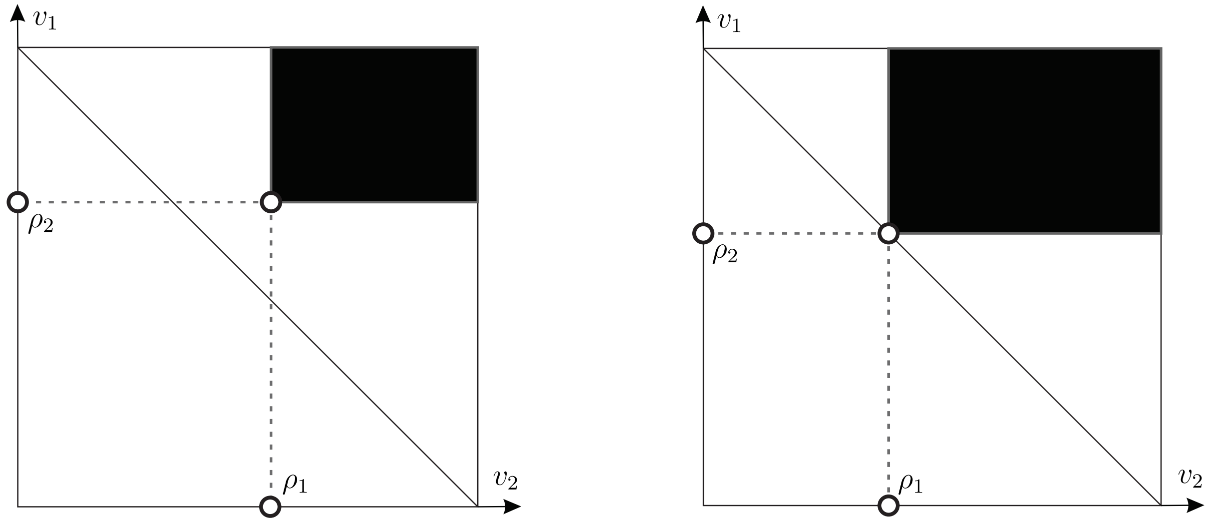

Figure 2: Allocation function of a *simple posted price* mechanism that runs budget surplus (left figure) and the one that is exactly budget balanced (right figure).

defined in Section 3.2. Because under Assumption 3, the transfers are uniquely defined by the allocation function $\varphi(v)$ of a robust mechanism, we only have to show that $\varphi(v)$ is actually a distribution function, or precisely, it satisfies the three characteristic properties in its definition. The first two properties follow immediately from Assumption 1 and Assumption 2, while the last property follows from Lemma 2. This establishes that $\varphi(v)$ is the *CDF* of some distribution over vectors $\rho \in V$ that are not yet restricted to the $\sum \rho_i = c$ hyperplane.

The last piece of the puzzle is the additional property in Lemma 2, that states that for any cube C that lies completely above or completely below that hyperplane, the $D_C^{1,\dots,n}$ operator is equal to zero. This means that the probability of a posted price p falling outside of the $\sum \rho_i = c$ hyperplane is zero, in other words they are all budget balanced. ■

Consider a functional $\mathcal{F}$ that maps an allocation function of the mechanism into a scalar. $\mathcal{F}$ is affine if $\mathcal{F} = k + \mathcal{L}$, where k is a constant and $\mathcal{L}$ is linear:

$$\mathcal{L}(\varphi_1 + \varphi_2) = \mathcal{L}(\varphi_1) + \mathcal{L}(\varphi_2), \quad \mathcal{L}(\lambda\varphi_1) = \lambda\mathcal{L}(\varphi).$$

The following statement follows immediately from the fact that the family of robust mechanisms is a convex hull of simple posted prices.

Corollary 1 *Any affine functional $\mathcal{F}$ attains minimum over the family of robust divisible mechanisms satisfying Assumptions 1,2,3 at a simple posted price, for any belief F.*

4.2 Optimality

Denote the welfare at the first best mechanism and the welfare at mechanism μ as:

$$W_{fb}(v) = (\sum v_i - c) \cdot \mathbb{I}[\sum v_i \geqslant c], \quad W_\mu(v) = (\sum v_i - c) \cdot \varphi_\mu(v),$$

Then denote the welfare loss function:

$$WL_\mu(v) = W_{fb}(v) - W_\mu(v)$$

The welfare loss works as an non-linear operator on the space of functions over V. It maps an allocation function $\varphi_\mu(v)$ of mechanism μ into a function $WL_\mu(v)$.

We can now define our main welfare criterion.

Definition 6 *The **expected welfare loss** $EWL_{F,\mu}$ is a function of a mechanism μ:*

$$EWL_{F,\mu} = \int WL_\mu(v)dF(v)$$

where F is the belief about the distribution of types.

Technically speaking, expected welfare loss is a non-linear functional acting on the space of allocation functions $\varphi_\mu(v)$. We will later show that the first one satisfies a certain strong property that we define below:

Definition 7 *A mechanism μ is **robustly optimal** for some prior beliefs F, if it minimizes expected welfare loss over a family of robust divisible mechanisms.*

Proposition 2 *For any prior belief F, a robustly optimal (among those satisfying Assumptions 1,2,3) mechanism is either a simple posted price, or a lottery over simple posted prices that yield the same expected welfare loss.*

Proof. By Corollary 1, it is sufficient to verify that, for any prior belief F, expected welfare loss of a robust mechanism μ is an affine functional acting on the space of robust allocation functions. This follows directly from formulas (2):

$$EWL_{F,\mu} = \int W_{fb}(v)dF(v) - \int W_\mu dF(v) =$$
$$= k(F) - \int \sum_i U_i(v)dF(v) = k(F) - \sum_i \int \int_{a_i}^{v_i} \varphi(z, v_{-i})dzdF(v)$$

where $k(F)$ is the first best expected utility. ∎

4.3 Welfare bounds

To derive the welfare bounds we introduce another supplementary welfare criterion.

Definition 8 *The **maximal welfare loss** MWL_μ is a function of a mechanism μ:*

$$MWL_\mu = \sup_v WL_\mu(v)$$

We will refer to the beliefs that maximize expected welfare loss at the corresponding optimal robust divisible mechanism as the *worst case beliefs*.

Proposition 3 *Expected welfare loss of a robustly optimal (among those satisfying Assumptions 1,2,3,4) mechanism at the worst case beliefs is bounded by:*

$$\frac{1}{4} \cdot (nb - c) \leqslant \sup_F \min_\mu EWL_{F,\mu} \leqslant \frac{n-1}{n} n^{\frac{-1}{n-1}} \cdot (nb - c)$$

Proof. To derive the lower bound, assume than n is even and split the agents in two equal groups: A, B. Fix a small $\varepsilon > 0$ and let F_ε be distributed on two tiny regions in V, as shown on Figure 3 for the special case of two agents. Precisely, let with probability

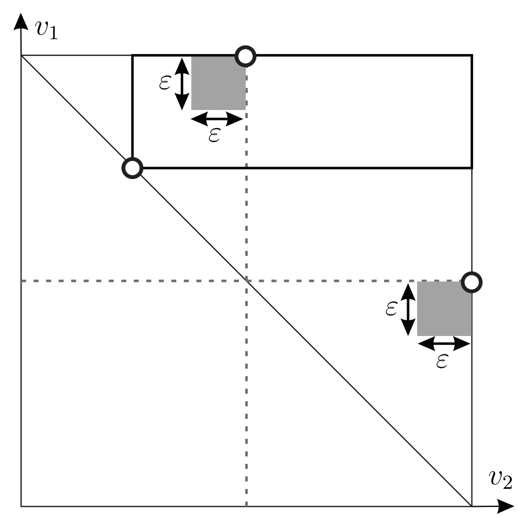

Figure 3: Distribution F_ε for $n = 2, V = [0, 1]^2$ and $c = 1$.

0.5 the valuations for agents in group A be iid uniformly on the segment $[c/n, c/n - \varepsilon]$ and the valuations for agents in group B be iid uniformly on the segment $[b, b - \varepsilon]$. Let with probability 0.5 the valuations be distributed similarly but with A and B switched places.

Since by Proposition 2, the optimal robust divisible mechanism is a posted price, it captures only one of the two regions and thus:

$$\lim_{\varepsilon \to 0} \min_\mu EWL_{F_\varepsilon,\mu} = 0.5 \cdot (\frac{n}{2}(b + \frac{c}{n}) - c) = \frac{1}{4}(nb - c)$$

If n is odd, let one agent always have the maximal valuation b and then repeat the procedure for the remaining $n - 1$ agents with costs $c - b$ to get the exact same number. This completes the construction of the lower bound.

To derive the upper bound, notice that by the standard max-min property:

$$\sup_F \min_\mu EWL_{F,\mu} \leqslant \min_\mu \sup_F EWL_{F,\mu} = \min_\mu MWL_\mu.$$

Next, consider a uniform distribution of posted prices on a special set $S \subset \mathbb{R}^n$:

$$S = \{\sum_i \rho_i = c,\ c - \sum_{j \neq i} v_j \leqslant \rho_i \leqslant b_i\}.$$

A corresponding divisible mechanism μ^* is robust, but it might not satisfy Assumption 3 because prices can be negative. The allocation function of this mechanism and the maximal welfare loss are easily calculated:

$$\varphi^*(v) = \frac{(\sum v_i - c)^{n-1}}{(nb - c)^{n-1}}, \quad MWL_{\mu^*} = \frac{n-1}{n} n^{\frac{-1}{n-1}} \cdot (nb - c)$$

Notice that this mechanism can be modified to satisfy Assumption 3 without increasing the maximal welfare loss. To do this it is sufficient to replace every lottery outcome ρ' that falls outside of the domain V with a lottery ρ'' on the boundary of V such that $\rho''_i \geqslant \rho'_i$. Intuitively, when a lottery offers negative prices to some of the agents, we replace them with zero and use the net profit to provide discounts to other agents. This completes the construction of the upper bound. ∎

From the Theorem above, we can immediately derive the following property.

Corollary 2 *The lower bound in Proposition 3 is tight for two agents.*

$$n = 2: \quad \sup_F \min_\mu EWL_{F,\mu} = \frac{1}{4}(nb - c).$$

Corollary 3 *For certain beliefs, the expected welfare loss of a robust divisible mechanism is as big as 1/2 of the expected welfare at the optimal Bayesian mechanism.*

Proof. Observe that under the sequence of beliefs F_ε defined in Proposition 3, the agents' types are perfectly correlated in the limit. This gives the designer ability to verify their reports in equilibrium and thus implement the first best allocation. Clearly, for ε arbitrarily close to zero, the welfare loss can be made arbitrarily small.

Since the total welfare under the beliefs F_ε is arbitrarily close to 1/2 of the maximal welfare, the result follows from Proposition 3. ∎

5 Discussion

Divisible mechanisms have a potential to perform better than binary ones, but at the same time are significantly harder to study. Unless some sort of reduction is made, constrained optimization over non-decreasing allocation functions does not have a clear solution. One such reduction is a characterization of divisible mechanisms in terms of lotteries over binary.

In this paper we pointed out the assumptions under which such reduction is valid in our linear setting. Once we show that robust mechanisms can be represented as randomizations over simple posted prices, optimization becomes extremely simple. Clearly, linearity of utilities and costs is critical to the validity of our results. An interesting avenue of research would be to identify optimal mechanisms without these strong assumptions.

We do not explicitly solve robustly optimal mechanism for any possible prior distribution of beliefs, but we show that such mechanism has to take the shape of a simple posted price. This gives foundation to a commonly used assumption of a binary allocation.

It is easy to see that under certain beliefs the expected welfare losses can be as small as zero. It therefore makes sense to try to identify the highest loss at a robustly optimal mechanism, over all possible beliefs. For the special case of two agents we have identified this loss to be exactly equal to $\frac{1}{4}(nb-c)$, that is, 25% of the maximal possible welfare, or 50% of the second-best welfare in this economy. For the multi-agent setting we show that the loss keeps growing at least proportionally to the size of the economy.

The question of whether there is a steady upper bound to the welfare losses is open, since we only provide a weak bound of $\frac{n-1}{n}n^{\frac{-1}{n-1}} \cdot (nb-c)$ that converges to $(nb-c)$ - the maximal possible welfare in the economy.

6 Appendix

Proof of Lemma 2.

To fill in the gap in the proof of Lemma 2 we will show that equation (6) implies that $D_C^{1,\dots,n}\varphi = 0$ for any cube $C \in \mathcal{C}$ that fully belongs to E, as long as the allocation function $\varphi(v)$ is non-decreasing and right-continuous is every variable.

The first step is to transform the equation (6) into a differential equation in an appropriate sense (see Evans (2010) textbook for 'weak differentiation' of generalized functions):

$$\int \dots \int \varphi(v) \left(\frac{\partial^n}{\partial v_1 \dots \partial v_n} \psi(v) \right) dv_1 \dots dv_n = 0 \tag{8}$$

which holds for any smooth function $\psi(v)$ with support in E.

Denote the following differentiation and integration operators:

$$\frac{\partial^n}{\partial v} = \frac{\partial^n}{\partial v_1 \dots \partial v_n}, \quad \frac{\partial^{n-1}}{\partial v_{-i}} = \frac{\partial^{n-1}}{\partial v_1 \dots \partial v_{i-1} \partial v_{i+1} \dots \partial v_n}$$
$$\int dv = \int \dots \int dv_1 \dots dv_n, \quad \int dv_{-i} = \int \dots \int dv_1 \dots dv_{i-1} dv_{i+1} \dots dv_n$$

Because $\psi(v)$ spans the same space of test functions as $(\sum_i v_i - c)\psi(v)$, let's write the desired equation as $J = 0$, where

$$J = \int \varphi(v) \frac{\partial^n}{\partial v} [(\sum_i v_i - c)\psi(v)] dv = \int \varphi(v) [(\sum_i v_i - c) \frac{\partial^n}{\partial v} \psi(v) + \sum_i \frac{\partial^{n-1}}{\partial v_{-i}} \psi(v)] dv.$$

Substituting $(\sum_i v_i - c)\varphi(v)$ from formula (6) and rearranging the order of integration we get the following:

$$J = \sum_i \int (A_i(v_{-i}) + B_i(v_{-i}))dv_{-i}$$

$$A_i(v_{-i}) = \int (\int_{c-\sum_{j\neq i} v_j}^{v_i} \varphi(z, v_{-i})dz)\frac{\partial^n}{\partial v}\psi(v)dv_i, \quad B_i(v_{-i}) = \int \varphi(v_i, v_{-i})\frac{\partial^{n-1}}{\partial v_{-i}}\psi(v)dv_i$$

then, applying integration by parts we get

$$A_i(v_{-i}) + B_i(v_{-i}) = [(\int_{c-\sum_{j\neq i} v_j}^{v_i} \varphi(z, v_{-i})dz)\frac{\partial^{n-1}}{\partial v_{-i}}\psi(v)]\Big|_{\underline{v}_i(v_{-i})}^{\overline{v}_i(v_{-i})}$$

where $\underline{v}_i(v_{-i}), \overline{v}_i(v_{-i})$ are boundaries of the support of $\psi(v)$ for a fixed level of v_{-i}. Finally, because $\psi(v)$ is a smooth test function, it has zero value at the boundary together with all of its derivatives, therefore $J = 0$ and the first step is complete.

The second step is to transform (8) into:

$$D_C^{1,\dots,n}\varphi = 0$$

that should hold for any cube $C \in \mathcal{C}$ such that $C \subset E$. To show that we will evaluate (8) at an indicator function

$$\hat{\psi}(v) = \mathbb{I}(v \in C), \quad C = \{v \in V \mid v_i \in (\tilde{a}_i, \tilde{b}_i], \ \forall i\}.$$

Note that since $\hat{\psi}(v)$ is not smooth, we can not do it directly.

Pick any smooth, symmetric kernel function $K(v)$ with support in $[0,1]^n$ and construct a sequence of smooth approximations to $\varphi(v)$:

$$\psi_n(v) = \int K_n(\eta)\hat{\psi}(v-\eta)d\eta, \quad K_n(\eta) = K(n\eta)$$

By construction this function is smooth with support in E (if n is big enough), and it's n-th mixed derivative is equal to zero everywhere except for the right-neighborhoods of the vertices of cube C. Because in these neighborhoods $\psi(v)$ is an indicator function, differentiating $\psi_n(v)$ simply recovers the smoothing kernel with alternating signs:

$$\frac{\partial^n}{\partial v}\psi_n(v) = \begin{cases} K_n(v-\tilde{b}), & \tilde{b}_j \leqslant v_j \leqslant \tilde{b}_j + 1/n \text{ for all } j \\ -K_n(v_i - \tilde{a}_i, v_{-i} - \tilde{b}_{-i}), & \tilde{a}_i \leqslant v_i \leqslant \tilde{a}_i + 1/n, \tilde{b}_j \leqslant v_j \leqslant \tilde{b}_j + 1/n \text{ for all } j \neq i \\ \dots & \\ (-1)^n K_n(v-\tilde{a}), & \tilde{a}_j \leqslant v_j \leqslant \tilde{a}_j + 1/n \text{ for all } j \\ 0, & \text{otherwise} \end{cases}$$

Finally, because the function $\varphi(v)$ is right-continuous in $(v_1, \dots, v_n)$ jointly, the integration over the smoothing kernel approximates the value at the vertices of the cube, with alternating signs. The whole integral is therefore an approximation to the sum of the values of $\varphi(v)$ at the vertices of the cube, with alternating signs:

$$\int \varphi(v) \frac{\partial^n}{\partial v} \psi_n(v) dv = D_C^{1,\dots,n}\varphi + o(1) \quad \Rightarrow \quad D_C^{1,\dots,n}\varphi = 0$$

This completes the proof.

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