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Robust Mechanism Design of Exchange

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In this article, we provide mechanisms for exchange economies with private information and interdependent values, which are ex post individually rational, incentive compatible, generate budget surplus, and are ex post nearly efficient, with many agents. Our framework is entirely prior-free, and we make no symmetry restrictions. The mechanisms can be implemented using a novel *discriminatory conditional double auction*, without knowledge of information structure or utility functions. We also show that no other mechanism satisfying the constraints can generate inefficiency of smaller order.

Key words: Exchange economy, Asymptotic efficiency, Interdependent values, Prior-free mechanisms, Robustness.

JEL Codes: D44, D47, D82

1. INTRODUCTION

The design of efficient trading mechanisms for exchange economies is one of the central problems in economics. In a complete-information environment, one way to achieve efficiency is to post Walrasian prices and let agents buy and sell freely. The problem, however, is harder if agents have private information, especially when values are interdependent, as in the case of asset trade. For instance, if agents trade using a double auction, which mimics the protocol on centralized stock, bond, or commodity exchanges, the equilibria are sensitive to the information structure and typically inefficient, even with many traders (see Rostek and Weretka, 2012). Indeed, in this setting, no feasible mechanism that guarantees a near-efficient allocation in a prior-free way is known.

In a private information environment, the prior-free approach that we adopt guarantees robustness of the mechanisms in the following sense (see Wilson, 1987; Bergemann and Morris, 2005; Chung and Ely, 2007). Bayesian approach presumes designer's knowledge of a joint distribution of agents' payoff types, beliefs about payoff types, beliefs about beliefs, and so on, and builds a different mechanism for each. Prior-free approach is more demanding in that it asks for *one* mechanism that works for *all* such distributions. We require that it satisfies the incentive

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constraints and is near-efficient ex post, for any realization of payoff types. This requirement is motivated by practical considerations. The prior-free mechanism can be implemented by an algorithm, much like ones executing centralized limit-order books on modern exchanges. It does not require an informed social planner. The goal of this article is to provide such mechanisms. It has the following three results. First, we construct a class of direct mechanisms for exchange economies with private information and interdependent values, with no symmetry restrictions, that are ex post individually rational, incentive compatible, and generate budget surplus for any economy size. Moreover, they result in an ex post nearly efficient allocation, when there are many, appropriately "small" agents. Second, the order of the inefficiency of the mechanisms is tight and cannot be improved by other mechanisms satisfying the constraints. Third, we show that the mechanisms can be implemented using a novel *discriminatory conditional double auction*. The auction can be run without any knowledge of either information structure or utility functions.

We motivate our results using the following stylized example (see Rostek and Weretka, 2012 and Examples 2, 5, and 7). Consider an exchange economy with a single asset and two large groups of agents, say, "fundamental" and "liquidity" traders. The value of a fundamental trader depends on his signal and the average of other fundamental traders' signals.[1] How does one design a trade mechanism that achieves a near-efficient allocation? One may try a double auction: let each agent submit a demand schedule specifying how much he wants to buy or sell at any price, and have the designer choose the price that clears the market. Aside from the Bayesian formulation, this game will not lead to a near-efficient allocation, regardless of the size of the economy. The problem is that a single price cannot convey all the relevant public information needed to make the right choice (Rostek and Weretka, 2012). For a fundamental trader, a high price might be due to high demand by the liquidity traders, indicating a perfect selling opportunity, or due to high demand by other fundamental traders, indicating high own value. Without enough information, he will not be able to trade near-efficient quantity.

A different mechanism would elicit private information via appropriate scoring rules, much in the spirit of Cremer and McLean (1985, 1988), and then use it to set the Walrasian price. The latter approach, however, works only if the designer knows the distribution of signals.[2] Finally, one could try to use prior-free direct *Vickrey–Clark–Groves (VCG)* mechanisms, but in exchange environments they run a budget deficit, and so are infeasible. The mechanisms we define in this article provide an alternative, as we explain below.

In this article, we look at exchange economies with N agents that have quasilinear utilities over a single divisible good and money. Each agent has a one-dimensional payoff type, and his utility depends on the whole profile of types (*interdependent values*). We make two important assumptions. First, the slopes of the individual demand curves are bounded. Second, the average effect of any single agent's type on marginal utilities of others converges to zero as the economy grows (*small pairwise interdependence*). In the fundamental/liquidity traders example, pairwise interdependence is of order $\frac{1}{N}$. Both assumptions formulate the notion of "small" agents in our environment, and guarantee a decentralized, competitive economy when N is large. As we show, they are sufficient and necessary for the near-efficiency in large economies.

1. It is irrelevant for the argument whether the liquidity traders have private or interdependent values.

2. Scoring rules may extract the agent's beliefs about others. However, exact knowledge of the joint distribution is needed to map such beliefs to the agent's payoff type; see Neeman (2004).

The main result of the article is a construction of *Vickrey–Clarke–Groves mechanisms with σ-quadratic taxes (σ-VCG)*, which are direct mechanisms that satisfy all the constraints ex post. Moreover, they are almost efficient ex post, when many agents are present (see Section 3). For an economy of size N, the maximal distance to the efficient allocation, across all agents and type profiles, is proportional to the sum of $\frac{1}{N}$ and the bound on pairwise interdependence. The mechanisms also guarantee a vanishing deadweight loss, when pairwise interdependence is of order at most $\sqrt{\frac{1}{N}}$. More generally, σ-VCG provide a linear trade-off between inefficiency and budget deficit.

σ-VCG mechanisms are based on allocations that introduce a wedge between the market price and each agent's marginal utility, and the wedge is linear in the quantity traded, with slope σ. It can be thought of as a controlled version of demand reduction, familiar from the multi-unit auction literature (Vives, 2011; Ausubel et al., 2014). In particular, no one is excluded from trade with any other agents (as in McAfee (1992), Córdoba and Hammond (1998), Tatur (2005), Loertscher and Mezzetti (2016), Kojima and Yamashita (2016), Loertscher and Marx (2017)). Instead, relying on divisibility of the good, each agent's trade is scaled down, thus enabling the mechanisms to achieve vanishing inefficiency ex post, and not just in expectation.

The second main result of the article shows that the above positive result is tight: any mechanisms satisfying the constraints must result in efficiency losses proportional to those achieved by appropriate σ-VCG mechanisms, for any size of the economy. The idea is that in any incentive compatible mechanism, agents must collect information rents for each inframarginal unit traded, with the highest rents for the "most inframarginal" units. To prevent budget deficit, mechanisms must collect enough extra funds to cover the rents. The linear wedge in our mechanism corresponds to a linear per-unit tax on each inframarginal unit traded. When the slope of the tax is comparable to the rate of change in the information rent, funds collected cover rents ex post with little surplus, and so cause minimal distortion. Moreover, for smaller slopes, the mechanisms trace out a "Pareto frontier" of optimal pairs of budget deficits and inefficiencies for any economy size, up to a multiplicative constant.

In the article, we also construct alternative mechanisms that are based on different ways of collecting extra funds. Examples include a fixed bid-ask spread (constant per-unit wedge) or a fixed entry fee (no wedge for sufficiently large trade). We show that among such mechanisms, linear wedges are strictly optimal. We also show that our results extend to the case in which goods are discrete, when taxes include a quadratic part together with an appropriate bid-ask spread.

σ-VCG mechanisms are direct mechanisms, in which the designer knows how to pick the allocation and transfers, given reported types. The third main result of the article is to provide an implementation using a *σ-Discriminatory Conditional Double Auction* (σ-DCDA), which, in the spirit of Wilson (1987) or Dasgupta and Maskin (2000), requires no such knowledge. In the game, each player submits his inverse demand, which specifies marginal prices at which he is willing to clear the market, conditional on not just a total quantity, but also on a vector of quantities that the other agents trade. The mechanism computes the market clearing quantity vector and price, as well as, for every trader, the residual demand curve that he is facing. Transfers are discriminatory: each trader pays the area under his residual demand curve, together with a tax quadratic in quantity. We show that an ex post equilibrium of this game exists that implements the σ-VCG allocation. In particular, when taxes are zero, the auction implements the VCG direct mechanism—and so the efficient allocation, at a deficit.

σ-DCDA is characterized by three features, each of which relates to practical problems in market design. First, ample evidence suggests that, in practice, market participants spend significant resources trying to identify a source of trade. Learning the source of trade from market data by high-frequency traders is a recent, well-publicized example.[3] Conditioning on the whole allocated vector in a σ-DCDA is an instrument that mitigates precisely this information flow. It allows each trader to infer information about the trading behaviour on the market directly as part of the trading protocol. Allowing retail investors to identify themselves as such on NYSE via the Retail Liquidity Program can be interpreted as facilitating a particular kind of conditional bidding.

High-dimensionality of bids in a σ-DCDA stems from the generality of our approach and is likely not a practical obstacle. At the extreme, when values are private, bids are conditioned only on the total quantity cleared just as in a standard double auction. We show that when utilities depend on a low-dimensional statistic of types, the equilibrium strategies are contingent on a low-dimensional statistic of the allocated vector. In the fundamental/liquidity traders example, fundamental traders submit prices conditional on the total quantity to be cleared and the average trade of other fundamental traders, with price increasing in each variable.

Second, quadratic taxes used in a σ-DCDA are an alternative to the constant ("fixed fee") or linear ("bid-ask spread") taxes, or fees, used in practice. They are designed to make fees proportional to the price impact that the traders have, and guarantee distortions of minimal order, while imposing no deficit ex post. We note that one way to interpret the weaker constraint of no deficit in expectation is that it requires an informed and financially unconstrained market maker, who facilitates trade and breaks even in expectation.[4] Our mechanisms do not require such an institution and can be implemented by a computer algorithm (see Section 5).

Finally, we show that appropriate discriminatory pricing obviates any form of strategizing and endogenous demand reduction, with information rents expressed directly in terms of lower inframarginal prices. This resembles the way second-price auctions eliminate incentives for shading in the first-price auctions, in the classical auction theory. Indeed, in equilibrium, agents use "revelation strategies" submitting marginal utility adjusted by the linear wedge. The computation of equilibrium strategies is relatively easy, even in cases when linear strategies in the uniform-price double auction are non-analytical (see Section 4). Although harder to quantify than efficiency gain, we consider this strategic simplicity a crucial virtue of our mechanisms.

Relationship to the literature. Our article contributes to the literature on strategic foundations of Walrasian Equilibrium in large exchange economies. One strand of it focuses on the question of when double auctions approximate efficiency in Bayesian environment. Important contributions include Wilson (1985), Palfrey and Srivastava (1986), Postlewaite and Schmeidler (1986), Klemperer and Meyer (1989), Gresik and Satterthwaite (1989), Satterthwaite and Williams (1989), Rustichini et al. (1994), Fudenberg et al. (2007), and Cripps and Swinkels (2006), as well as Yoon (2001) and Tatur (2005) for double auctions with entry fees, in the case of private values, and Hellwig (1980), Mas-Colell and Vives (1993), Reny and Perry (2006), and Vives (2011) for interdependent values. In our article, we require stronger ex post constraints. Moreover, double auctions guarantee asymptotic efficiency only with symmetric agents (and so not in the fundamental/liquidity traders example; see Rostek and Weretka, 2012) and under additional assumptions on the distribution of signals, such as conditional independence. Finally, we stress that our mechanisms can be used for any economy size.

3. See the Concept Release on Equity Market Structure by the Securities and Exchange Commission (2010) or, in more colourful language, Lewis (2014). The rapidly growing literature on HFT is surveyed in Jones (2013), Biais and Foucault (2014), and the Securities and Exchange Commission (2014).

4. Of course, real-life market makers are not interested in efficiency but profit maximization. Near efficiency requires an additional assumption of competition between such idealized market makers.

Aside from double auctions, Roberts and Postlewaite (1976), Jackson (1992), and Jackson and Manelli (1997) show that Walrasian mechanisms are difficult to manipulate in large replica economies. Gul and Postlewaite (1992) and McLean and Postlewaite (2002) identify, in the Bayesian framework, crucial information-smallness requirements, and McLean and Postlewaite (2015) show that the benefits of ex post deviations in the efficient mechanism are small. We rely on related smallness assumptions but build mechanisms that satisfy exact ex post constraints and provide an auction implementation.

In the standard auction context, under common knowledge of the benefits of trade, VCG mechanisms generate budget surplus. In the private-value case, the mechanism is implemented by the second-price auction. Ausubel (1999), Dasgupta and Maskin (2000), Eso and Maskin (2002), Perry and Reny (2002, 2005), Ausubel (2004), and Izmalkov (2004) extend and show how to implement VCG mechanisms with interdependent values and multiple goods. With taxes set to zero, our discriminatory conditional double auction provides a new implementation of the efficient (unbalanced) VCG in the exchange setting with interdependencies. The conditional bids in our implementation are related to the conditional bids in Dasgupta and Maskin (2000) and Eso and Maskin (2002). In our case, inverse demands condition on vectors of allocated quantities, whereas their demands condition on vectors of utility functions. Contingent demands are also used in a context of dynamic trade with private values by Sannikov and Skrzypacz (2016) and Du and Zhu (2017).

Few studies have offered mechanisms for exchange environments that satisfy ex post constraints and are almost efficient in expectation, when the economy is large. In a seminal paper, McAfee (1992) provides a simple mechanism and an auction implementation that achieve this with independent private values and unit demands and supplies, whereas Huang et al. (2002), Chu and Shen (2008), and Loertscher and Mezzetti (2016) deal with multiunit trade.[5,6] Kojima and Yamashita (2016) allow for interdependent values when agents are symmetric and types are conditionally independently distributed. Our framework has interdependent values with no symmetry restrictions. Crucially, σ-VCG mechanisms are almost efficient not just in expectation, but ex post.

The remainder of the article is organized as follows. Section 2 describes the setup and establishes the lower bound on the inefficiency/budget deficit of mechanisms (Proposition 1). Section 3 defines the σ-VCG direct mechanisms and shows that they achieve inefficiency/budget deficit proportional to the lower bound (Proposition 2). Section 4 illustrates σ-VCG mechanisms in different economies and compares them with double auctions. σ-DCDA auctions are defined in Section 5, and Proposition 3 shows that they implement σ-VCG mechanisms. Other mechanisms are considered in Section 6, and Section 7 provides a discussion. The proofs and the extension to a discrete setting are in Appendix.

2. SETUP

2.1. *Payoffs*

An exchange economy consists of N agents who can trade a single good, or asset, for money. Each agent i has a payoff type $s_i \in \mathbb{R}$ and a quasilinear utility function

$$U_i(q_i, t_i, s) = u_i(q_i, s) - t_i,$$

5. The mechanism in McAfee (1992) also guarantees bounded ex post inefficiency, as the economy grows; see Chu (2009) for a similar result in a multi-unit setting.

6. See Baliga and Vohra (2003) and Segal (2003), among others, for robust monopoly pricing.

where $q_i \in \mathbb{R}$ is the quantity of the good that agent i gets, t_i is the transfer that he makes, and $s=(s_1,...,s_N)\in S$ is a profile of payoff types.[7] We emphasize that utilities may depend on the whole vector of types (*interdependent values*) and that we are not making any symmetry assumptions.

The set of all payoff type profiles $S \in \mathbb{R}^N$ need not be a product set, but we assume for simplicity that for every s_{-i}, the projection $\{s_i | (s_i, s_{-i}) \in S\}$ is an interval.[8] Functions u_i are strictly concave in q_i and are twice continuously differentiable. We will write $q=(q_1,...,q_N)$, $t=(t_1,...,t_N)$ for the profiles of quantities and transfers, respectively, and $mu_i(q_i,s)=\frac{\partial}{\partial q_i}u_i(q_i,s)$, $mu_{i,j}(q_i,s)=\frac{\partial^2}{\partial q_i \partial s_j}u_i(q_i,s)$, and $mu_{i,q}(q_i,s)=\frac{\partial^2}{(\partial q_i)^2}u_i(q_i,s)$ for marginal utilities and their derivatives. We will assume that $mu_{i,j}(q_i,s)\geqslant 0$, for all $i,j\leqslant N$, q_i, and s.

The framework is a standard exchange economy setting with a single divisible good, and money. The efficient benchmark in this setting is, of course, the *Walrasian Equilibrium (WE) allocation profile* $\{q^0(s)\}_{s\in S}$. For any vector of types s, the allocation $q^0(s)$ is defined jointly with the *Walrasian price* $p^0(s)$ by

$$mu(q_i^0(s),s)=p^0(s), \;\; \forall i, \quad \sum_i q_i^0(s)=0. \tag{2.1}$$

A unique WE allocation exists for any type profile s (see Section 3), and, from strict concavity of the utility functions, any other allocation results in a positive deadweight loss.

For the main results, we need additional assumptions about the utility functions.

$$\begin{aligned}
&(A1) && \left|q_i^0(s)\right| \leqslant \overline{q}, \;\; \forall i,s.\\
&(A2) && mu_{i,q}(q_i,s)\in[-\overline{m}_q, -\underline{m}_q]<0, \;\; \forall i,q_i,s.\\
&(A3) && mu_{i,i}(q_i,s)\geqslant \underline{m}_o>0, \;\; \forall i,q_i,s.\\
&(A4) && \tfrac{1}{N-1}\textstyle\sum_{i\neq j} mu_{i,j}(q_i,s)\leqslant \phi_N, \;\; \text{with } \lim_{N\to\infty}\phi_N=0, \;\; \forall j,q_{-j},s.
\end{aligned}$$

All those assumptions are substantial. Formally, Proposition 1 establishes that each of the bounds is necessary for near-efficiency in large markets. Informally, we argue below that they capture the right notion of "smallness" of agents in our environment, and guarantee that the economy is decentralized, or competitive, and not dominated by a single or few traders. If an agent has a large impact on the economy and can control prices, efficiency is out of reach.

Assumption *A1* bounds the size of the efficient trade. Without this assumption, in any candidate mechanism, one "big" trader could affect the market clearing price and allocations for many other, "small" traders, by choosing to trade either a lot or nothing.[9]

Assumption *A2* bounds the slope of the individual demand curves, in the case when information is public. Otherwise, say, if all but one agents had steep demand curves, their aggregate demand curve would have a moderate slope. In other words, the aggregate of such myriad agents would behave as one, leaving significant power to determine price and quantity in the hands of the single "large" trader they face.

We note that *A2* is violated in a model with discrete allocations. Even though the allocations can be convexified using lotteries, the marginal utility of each agent drops discontinuously at discrete units. Indeed, as before, no matter how many agents are in the economy, the aggregate demand

7. Normalizing the initial endowment to zero is purely for notational convenience.

8. This assumption is not essential for the results, but simplifies the incentivizing transfers in the mechanisms that we consider—see (3.10).

9. The assumption is needed only to bound efficiency losses: without it, mechanisms we construct satisfy all other constraints, particularly budget surplus (see Proposition 2).

curve may be vertical for prices between the common grid of marginal utilities. In Appendix G, we show, however, that our main results extend to the case of discrete allocations, when Assumption *A*2 is replaced by one that bounds the change in marginal utilities for consecutive units.

Finally, Assumptions *A*3 and *A*4 jointly mean that as the economy grows, the average impact of each agent's information on marginal utilities of others becomes relatively insignificant.[10] Otherwise, regardless of the economy size, a fixed small group of traders could monopolize the relevant information about the common-value component of the asset. Each of those "informationally large" traders has a large impact on the price, and the incentive problem is similar as if only they were present. Near-efficiency is out of reach (see Myerson and Satterthwaite, 1983). We refer to *A*4 as the assumption of *small pairwise interdependence.*

We note that Assumptions *A*3 and *A*4 do not restrict the models we consider to small perturbations of the private value model. Agents' utilities may be virtually determined by public shocks (see Examples 1 to 3). The only restriction is that the information about those public shocks is dispersed.

Working with economies of fixed size N, we use the following counterpart of *A*4:

$$(A4^N) \qquad \frac{1}{N-1}\sum_{i\neq j} mu_{i,j}(q_i,s) \leqslant \phi_N, \text{ with } \phi_N < \min\left\{\frac{\underline{m}_q}{\overline{\overline{m}}_q}, \frac{N-2}{2(N-1)}\right\}\underline{m}_o, \quad \forall j, q_{-j}, s.$$

In particular, we allow the single crossing assumption to be violated, and thus allow $j's$ signal to be more relevant to some other trader, $mu_{i,j}(q_i,s) > m_{j,j}(q_j,s)$. $A4^N$ is weaker, in that it bounds only the average effect of $j's$ signal on others, but the bound, which is the impact on own utilities $\underline{m}_o$, is scaled down by $\min\left\{\frac{\underline{m}_q}{\overline{\overline{m}}_q}, \frac{N-2}{2(N-1)}\right\}$. The first scaling factor $\frac{\underline{m}_q}{\overline{\overline{m}}_q}$ is required for monotonicity when dealing with only average effects (see Section 7). The second scaling factor $\frac{N-2}{2(N-1)}$ is necessary for the mechanism we construct to generate budget surplus (see (3.13) in Proposition 2; see also Rostek and Weretka (2012) for similar bounds on interdependence needed for the existence of linear equilibria in double auctions). It is above $\frac{1}{4}$ for every economy size $N > 3$, and increases—relaxing the constraint—as the economy grows.

2.2. *Motivating examples*

Example 1 *(Fundamental value model, Vives (2011), Rostek and Weretka (2012)) Utilities are*

$$u_i(q_i,s) = (\alpha s_i + \beta \bar{s})q_i - \frac{\mu}{2}q_i^2,$$

*for some constants $\alpha, \mu > 0$, and $\beta \geqslant 0$, where $\bar{s}$ is the arithmetic average of the types. Each agent is uncertain about the value of the first infinitesimal unit of the good or the intercept of the linear marginal utility function. The value is a weighted average of common and idiosyncratic shocks, $\bar{s}$ and s_i. It may reflect both the expected cash flow of the asset, common to all the agents that are partially informed about it, and private hedging needs. We have $mu_{i,j}(q_i,s) = \frac{\beta}{N}$ for any $j \neq i$, and so the small pairwise interdependence Assumption A*4 *is satisfied; for fixed N, $A4^N$ holds as long as $\alpha > \frac{\beta}{N-2}$. The efficient WE allocation is*

$$q_i^0(s) = \frac{\alpha}{\mu}(s_i - \bar{s}). \tag{2.2}$$

10. Assumption *A*4 normalizes the signals. Without it, Assumption *A*4 has no bite and can be assumed in any family of models, given a straightforward change of variables.

Example 2 *(Group model, Rostek and Weretka (2012)) Suppose agents are divided into two groups of the same size, and the utilities are*

$$u_i(q_i,t_i,s)=(\alpha_G s_i+\beta_G \overline{s}^G)q_i-\frac{\mu}{2}q_i^2,\ \ \forall i\in G, G=1,2,$$

where $\overline{s}^1$ *and* $\overline{s}^2$ *are the average types in each group,* $\alpha_1,\alpha_2,\mu>0$ *and* $\beta_1,\beta_2\geqslant 0$. *Compared to Example 1, in addition to the idiosyncratic shocks, two shocks now exist,* $\overline{s}^1$ *and* $\overline{s}^2$, *common to the respective group and irrelevant to the other. Agents are trading either for fundamental or liquidity reasons, and values are correlated within a group. The efficient WE allocation is*

$$q_i^0(s)=\frac{\alpha_G s_i+\beta_G \overline{s}^G-(\alpha_G+\beta_G)\overline{s}}{\mu},\ \ \forall i\in G, G=1,2. \tag{2.3}$$

Example 3 *(Fundamental value model with heterogeneous traders) Two groups of agents exist, "small" and "big," and utilities are*

$$u_i(q_i,s)=(\alpha s_i+\beta\overline{s})q_i-\frac{\mu_G}{2}q_i^2,\ \ \forall i\in G, G=S,B,$$

where $\mu_S=\overline{\mu}$, $\mu_B=\underline{\mu}$, *and* $\overline{\mu}>\underline{\mu}$. *As in the Fundamental Value Model, all the agents are partially and symmetrically informed about the fundamental value of the risky asset, but they differ now in their capacity to hold a large position, with big agents relatively risk neutral (or bearing a large cost of staying away from their ideal position). The efficient WE allocation is*

$$q_i^0(s)=\frac{\alpha s_i+\beta\overline{s}-p^\sigma(s)}{\mu_G},\ \ \forall i\in G, G=S,B. \tag{2.4}$$

A natural source of interdependence in preferences is informational, when the agent's type corresponds to a signal informative of all agents' values of the asset. We have used this interpretation in the examples above and continue to do so throughout the article. This informational interpretation can be further justified via Bayesian models. Indeed, suppose each agent i has a private state space Θ_i and a continuously differentiable utility function $v_i(q_i,\theta_i)$ as well as a prior $\delta_i\in\Delta(\Theta_i\times S_1\times\ldots\times S_N)$. If one defines

$$u_i(q_i,s)=\mathbb{E}_{\delta_i}[v_i(q_i,\theta_i)|s],$$

the correlations of signal vectors s and value θ_i translate to utilities u_i with interdependent values.

In particular, utility functions in the examples above are expected utilities in the model in which each agent i has linear-quadratic utilities,

$$v_i(q_i,\theta_i)=\theta_i q_i-\frac{\mu}{2}q_i^2, \tag{2.5}$$

and believes $(\theta_i,s_1,\ldots,s_n)$ has a joint Normal distribution, where the linear parameters $w_j, j\leqslant N$, are such that

$$\mathbb{E}_{\delta_i}[\theta_i|s_1,\ldots,s_N]=\sum\nolimits_j w_j s_j, \tag{2.6}$$

from the Projection Theorem for Normal distributions. In other words, a fixed linear-quadratic utility function with parameters $w_j, j\leqslant N$ may be justified, for example, by any Normal prior beliefs for which (2.6) holds. We stress here that *many* beliefs may give rise to the *same* utilities

u_i; we discuss this Bayesian justification for the interdependent preferences in the context of robustness in Section 7 (see also Example 6).

Our notion of small pairwise interdependence is closely related to the informational smallness used in Bayesian models (see Gul and Postlewaite (1992) and McLean and Postlewaite (2002, 2015)). If Θ_i and $\frac{\partial mv_j}{\partial \theta_i}$ are bounded for all i,j, then u_i satisfy A4 precisely when[11,12]

$$A4^B) \qquad \frac{1}{N-1}\sum_{i\neq j}\left|\frac{\partial \delta_i(\theta_i|s)}{\partial s_j}\right| \leqslant \phi_N^B, \text{ with } \lim_{N\to\infty}\phi_N^B = 0, \ \forall j, \theta_{-j}, s.$$

In other words, given a Bayesian justification, the bound on pairwise interdependence A4 is equivalent to any single agent j having a small average impact on the beliefs of others about the payoff-relevant state.

2.3. *Mechanisms*

Given a profile of utility functions $(u_1,\ldots,u_N)$, a *(direct) mechanism* in our setting is $\{(q(s),t(s))\}_{s\in S}$, where $\{q(s)\}_{s\in S}$ is the *allocation profile*, $\{t(s)\}_{s\in S}$ is the *transfers profile*, and $q(s),t(s)\in\mathbb{R}^N$. A mechanism satisfies ex post: *market clearing (MC), δ-budget surplus (δ-BS), for $\delta\geqslant 0$, individual rationality (IR), incentive compatibility (IC)*, and *ε-efficiency (ε-Eff), for* $\varepsilon\geqslant 0$, if the following conditions hold:

$$\begin{array}{ll}
\text{(MC)} & \sum_i q_i(s)=0, \ \forall s. \\
(\delta\text{-BS}) & \sum_i t_i(s)\geqslant -\delta, \ \forall s. \\
\text{(IR)} & U_i(q_i(s),t_i(s),s)\geqslant U_i(0,0,s), \ \forall i,s. \\
\text{(IC)} & U_i(q_i(s),t_i(s),s)\geqslant U_i(q_i(s_i',s_{-i}),t_i(s_i',s_{-i}),s), \ \forall i,s,s_i'. \\
(\varepsilon\text{-Eff}) & \left|q_i(s)-q_i^0(s)\right|\leqslant\varepsilon, \ \forall i,s.
\end{array}$$

For δ and ε equal zero, we will call the respective constraints simply "BS" and "Eff."Finally, consider a family of mechanisms $\{(q(s),t(s))\}_{s\in S}$ for any number of players N and any utility profiles $(u_1,\ldots,u_N)$. For a function $f:\mathbb{N}\to\mathbb{R}_+$, we say the family is *robustly asymptotically f-efficient* if each mechanism satisfies the constraints MC, BS, IR, IC, and f_N-Eff, for the appropriate N.

Market clearing is a standard feasibility constraint. Budget surplus is similar, because we want mechanisms to run without an outside source of money; positive δ allows for some slack, with subsidies from the designer bounded by δ.[13] Individual rationality requires participation in a mechanism to be voluntary, and incentive compatibility means that truthful reporting of own type is optimal.

Crucially, all the constraints and the objective are required to be satisfied ex post for any vector of types. Although ex post constraints are natural in the case of market clearing, those requirements

11. We identify a distribution δ_i with its *cdf*.

12. The main difference is that McLean and Postlewaite (2002) require the bound in $A4^B$ to hold not for every s_{-j}, but only for s_{-j} with probability $1-\phi_N^B$. The strengthening in our article is dictated by the stronger, ex post version of incentive compatibility.

13. Requiring strictly positive surplus, and so $\delta<0$, would clash with individual rationality. This is because for typical utility functions, there exists a profile of types with no trade at the efficient allocation. Consequently, individual rationality implies that no positive revenue can be squeezed out of the agents.

are strong in the case of IR, IC, BS, and efficiency, relative to their Bayesian counterparts. All are motivated by practical considerations. Specifically, the benefits of the ex post constraints are as follows.

First, ex post IR and IC imply Bayesian IR and IC for any type space describing agents' beliefs—about types, about the beliefs of others, and so on (see *e.g.* Bergemann and Morris, 2005). Thus, ex post IR and IC guarantee that the mechanism is *robust with respect to the misspecification of agents' beliefs*. For practical purposes, the designer does not need to know what agents believe about others, when designing a mechanism.

Second, ex post IR and IC also guarantee that agents have *no regret* ex post about their allocation. The mechanisms will work in a weak contractual environment, when each agent can walk away from the transaction (or change their action) at any stage, even after observing the full allocation vector and possibly inferring types of other agents. This shares many of the features of a standard spot market. Relatedly, ex post constraints nullify any benefits of (inefficient) spying on other agents to infer their types (Bergemann and Välimäki, 2002).

Third, ex post BS and (near-) efficiency guarantee that BS and (near-) efficiency will be satisfied for any distribution of types. Thus, a mechanism is *robust with respect to the misspecification of type distribution,* and so the mechanism designer need not know it. Intuitively, Bayesian formulation requires an idealized—informed and financially unconstrained—market maker that facilitates trade. If the market maker knows the distribution of types, is risk neutral, and faces no liquidity constraints, he could broker the trade, while breaking even and achieving near-efficiency in expectation. Given the volume of trade and the number of assets traded on centralized exchanges, however, the informational and liquidity requirements of such risk-neutral brokering are staggering. By contrast, an ex post BS and (near-) efficient mechanisms can be implemented by a computer algorithm (see Section 5). What we have in mind is a counterpart of an algorithm that executes a centralized limit-order book on modern exchanges.

We note that, similar to most of the literature (see *e.g.* McAfee, 1992; Kojima and Yamashita, 2016), our notion of efficiency does not penalize for budget surpluses. One interpretation is that extra money need not be flushed down the drain, but can be used elsewhere. On the other hand, as long as marginal utilities are bounded, spelling out ε-efficiency directly in terms of differences in allocations and not utilities is clearly without loss of generality.[14] Moreover, ε-efficient mechanisms generate deadweight losses of order $\varepsilon^2 N$.[15]

Lemma 1 *Suppose* $|mu_i(0,s)| \leqslant m$, *for all* i,s. *If a mechanism* $\{(q(s),t(s))\}_{s\in S}$ *is* ε*-efficient and satisfies MC, then for every* s,

$$\left|u_i(q_i(s),s)-u_i(q_i^0(s),s,)\right| \leqslant \varepsilon m,$$

$$\varepsilon^2\frac{N\underline{m}_q}{2} \leqslant \left|\sum_{i=1}^N u_i(q_i(s),s)-\sum_{i=1}^N u_i(q_i^0(s),s)\right| \leqslant \varepsilon^2\frac{N\overline{m}_q}{2}.$$

The following proposition sets the stage for the main results in the following sections.

14. It trivially follows from Proposition 1 below that without a bound on marginal utilities, no mechanism that satisfies all other constraints can avoid infinite inefficiencies, spelled out in terms of utilities.

15. The second part of the lemma follows from *A2* and

$$\left|\sum_{i=1}^N u_i(q_i(s),s)-\sum_{i=1}^N u_i(q_i^0(s),s)\right| = \left|\int_{q_i^0(s)}^{q_i(s)} \left(mu_i(x,s)-p^0(s)\right)dx\right|.$$

Proposition 1 *Suppose Assumptions A1, A2, A3, and A4 hold. For appropriate $C>0$, no robustly asymptotically $C(\frac{1}{N}+\phi_N)$-efficient family of mechanisms exists.*

More precisely, for every $N>2$, there are economies of size N that satisfy A1, A2, A3, and $A4^N$, such that for every $\varepsilon \leqslant \overline{q}/3$, no mechanism satisfies MC, IR, IC, δ-BS, and ε-Eff, where

$$\delta = N\left[C_1^N\left(\frac{\underline{m}_o}{N-1}+\phi_N\right)-C_2\varepsilon\right], \tag{2.7}$$

$$C_1^N=\frac{1}{2}\left(\frac{N-1}{N}\overline{q}-\varepsilon\right)^2\frac{\overline{m}_q}{\underline{m}_o-\phi_N},\quad C_2=\overline{q}\times\overline{m}_q.$$

The result implies that, for a fixed economy size, no mechanism can guarantee robust implementation of the efficient WE allocation. Versions of such impossibility in an exchange environment are well known, and go back to the seminal contribution by Myerson and Satterthwaite (1983). However, the proposition also provides a lower bound on the extent to which the constraints may be violated, which will be the benchmark for the positive result in Proposition 2. Specifically, the proposition traces out a linear curve of budget-deficit and efficiency-loss pairs (δ,ε), such that no mechanism that satisfies IR, IC, and MC can guarantee both deficit below δ and inefficiency below ε.

On one extreme, full efficiency ($\varepsilon=0$) may require a budget deficit of at least $NC_1^N\times(\underline{m}_o/(N-1)+\phi_N)$, with constants C_1^N bounded, and bounded away from zero. On the other extreme, the problem of patching up the deficit can be "decentralized," with traders themselves providing the necessary funds. The result shows that requiring no budget deficit ($\delta=0$) may compromise efficiency by at least $C_1^N/C_2\times(\underline{m}_o/(N-1)+\phi_N)$.

Let us provide the intuition for the constants that determine the magnitudes of the deficit and the efficiency loss. We focus first on the budget losses of mechanisms that achieve the efficient WE allocation profile $\{q^0(s)\}_{s\in S}$, when ε is zero. Efficient allocation and incentive compatibility pin down the transfers, following the logic that dates back to Vickrey (1961). In our case, fix a type vector s and a buyer i, $q_i^0(s)>0$. To achieve incentive compatibility, the price that i pays for every $x'th$ inframarginal unit of the good, $x<q_i^0(s)$, must equal his value for this unit had he reported the type that makes him pivotal for it. Agent i is pivotal for $x'th$ unit of the good precisely when he reports the type $s_i^0(x)$ such that he gets exactly x, $q_i^0(s_i^0(x),s_{-i})=x$, and so his value for it equals the Walrasian price, $p^0\left(s_i^0(x),s_{-i}\right)$.

Each buyer i thus pays the integral of Walrasian prices $p^0\left(s_i^0(x),s_{-i}\right)$, across all the inframarginal units x, and the price is increasing in units purchased (for sellers: the price is decreasing in units sold). Because only the price for the marginal unit equals $p^0(s)$, the per-player contribution to the deficit equals at least the area of the triangle, which is half times the squared quantity traded, times the minimal speed with which the Walrasian price decreases in buyer $i's$ demand, $\frac{\partial p^0}{\partial q_i}$.[16]

In a model with constant maximal demand slopes $\overline{m}_q$, a minimal impact of own signals $\underline{m}_o$, and maximal levels of interdependence ϕ_N, the derivative $\frac{\partial p^0}{\partial q_i}$ is $\frac{\overline{m}_q}{\underline{m}_o-\phi_N}\left(\frac{\underline{m}_o}{N-1}+\phi_N\right)$. It equals the effect of $i's$ type on the Walrasian price, which is the average of the effects of $i's$ type on marginal utilities of the agents, $\frac{\partial p^0}{\partial s_i}=\frac{\underline{m}_o}{N}+\frac{N-1}{N}\phi_N$, divided by the slope of the allocation function, $\frac{\partial q_i}{\partial s_i}$.

16. In Figures 1 and 2, the contribution to the deficit equals the area below the Walrasian price $p^0(s)$ and above the dotted line, on the left panels. Areas above $p^0(s)$ and below the dotted lines on the right panels are the corresponding contributions of the seller to the deficit.

For an appropriate type profile, the quantity traded by every player must be above $\frac{N-1}{N}\overline{q}-\varepsilon$.[17] Overall, this argument justifies the δ in (2.7), when $\varepsilon=0$.

The first-order effect of compromising efficiency on decreasing the deficit is as follows. A buyer must be allocated at least the efficient level of units minus ε, for every signal profile, which implies that his marginal utility for each unit may be at most $\varepsilon\overline{m}_q$ above the corresponding Walrasian price (for the sellers that trade ε too little the price is $\varepsilon\overline{m}_q$ below the Walrasian price). This implies that the per-unit payments for those inframarginal units—following, again, the pivotal type logic—are at most $\varepsilon\overline{m}_q$ higher than in the case of the efficient mechanism. When multiplied by the maximum quantity traded, it provides an upper bound on the agent's contribution to the budget, as in the proposition.

3. VICKREY-CLARKE-GROVES MECHANISMS WITH σ-QUADRATIC TAXES

Proposition 1 says that no mechanism can achieve efficiency loss ε and budget deficit δ below the linear curve (2.7). In this section, in the main result of the article, we construct a class of mechanisms that achieve this linear bound, up to a multiplicative constant, for every economy. This construction is simplest in the case of full efficiency ($\varepsilon=0$), when conditions (2.1) pin down both the allocation and the incentivizing transfers. Other cases ($\varepsilon>0$) require a judicious choice of the allocation function, in every economy, so that the accompanying incentivizing transfers achieve a deficit proportional to (2.7).

Our results rely on the following mechanisms. We define the allocations and transfers separately.

Definition 1 ***(VCG mechanism with σ-quadratic taxes: allocation)*** *Fix a profile of utility functions $(u_1,...,u_N)$ and $\sigma\geqslant 0$. For any vector of types s, the allocation $q^\sigma(s)$ and price $p^\sigma(s)$ in a* Vickrey–Clarke–Groves with σ-quadratic taxes (σ-VCG) mechanism *are defined jointly via*

$$mu(q_i^\sigma(s),s)=p^\sigma(s)+\sigma\times q_i^\sigma(s),\ \ \forall i, \tag{3.8}$$
$$\sum_i q_i^\sigma(s)=0.$$

One may think of the σ-VCG allocation and price as the standard efficient WE allocation and Walrasian price, but for distorted utilities,

$$\widetilde{u}_i(q_i,s)=u_i(q_i,s)-\sigma\frac{q_i^2}{2}. \tag{3.9}$$

The second term is a "tax," which is quadratic in the quantity traded. Of course, 0-VCG (or, simply VCG) price is the Walrasian price, and the VCG allocation is the efficient WE allocation. Strictly positive σ, however, results in insufficient trade and compromises efficiency: a marginal utility of a buyer ($q_i^\sigma(s)>0$) is strictly greater than a marginal utility of a seller ($q_i^\sigma(s)<0$). This discrepancy in marginal utilities is related to demand reduction in the multi-unit auctions (see *e.g.* Ausubel et al., 2014; see Example 4).

17. The reason for the scaling constant $\frac{N-1}{N}$ is that a type profile at which the absolute value of efficient trade by each agent is exactly $\overline{q}$ is inconsistent with information rents. If, say, a buyer reported a lower signal, the efficient trade by other buyers would be above $\overline{q}$, violating $A1$. The further shift by ε is due to ε-Efficiency.

Given continuous differentiability and strict concavity of the utility functions, for any $\sigma \geqslant 0$, p, and $i \leqslant N$, at most one $q_i^\sigma(p,s)$ satisfies

$$mu_i(q_i^\sigma(p,s),s) = p + \sigma \times q_i^\sigma(p,s),$$

and the functions $q_i^\sigma(\cdot,s)$ are continuous and strictly decreasing in their domains. Therefore, at most one p exists for which $\sum_{i=1}^N q_i^\sigma(p,s) = 0$, which establishes uniqueness of $(p^\sigma(s), q^\sigma(s))$, for any $\sigma \geqslant 0$ and s. Existence follows from A2.

Definition 2 *(**VCG mechanism with σ-quadratic taxes: transfers**) Fix a profile of utility functions $(u_1,\ldots,u_N)$ and $\sigma \geqslant 0$. For any vector of types s, the transfers $t^\sigma(s)$ in a σ-VCG mechanism are defined as*

$$t_i^\sigma(s) = \int_0^{q_i^\sigma(s)} p^\sigma\left(s_i(x), s_{-i}\right) dx + \frac{\sigma}{2} q_i^\sigma(s)^2, \tag{3.10}$$

where $\{q^\sigma(s)\}_{s\in S}$ and $\{p^\sigma(s)\}_{s\in S}$ are the σ-VCG allocations and prices, and for any agent i and quantity x between 0 and $q_i^\sigma(s)$, $s_i(x)$ is the signal such that[18]

$$x = q_i^\sigma\left(s_i(x), s_{-i}\right), \tag{3.11}$$

$$s_i(x) = \begin{cases} \inf S_i \text{ if } q_i^\sigma\left(s_i', s_{-i}\right) > x, & \forall s_i', \\ \sup S_i \text{ if } q_i^\sigma\left(s_i', s_{-i}\right) < x, & \forall s_i'. \end{cases}$$

Transfers by any agent i consist of a discriminatory part and a tax. Tax is quadratic in own quantity traded or, alternatively, is a linear per-unit tax. It is a positive contribution by everyone. The discriminatory part is the integral over per-unit prices for each inframarginal unit traded, where price for unit x is the σ-VCG price under a counterfactual report $s_i(x)$ that results in i trading exactly x. Thus, it is simply the integral under the residual demand curve that i is facing.

In the case of private values, at any type profile, the discriminatory transfers by agent i are the integral under the aggregate linear tax-adjusted marginal utilities of agents other than i (see Figure 1). With interdependent values, the transfers are more discriminatory: say, a lower report by a buyer, which results in a lower quantity he trades, also decreases marginal utilities of other agents, and so further depresses the prices for inframarginal units (see Figure 2).

The following is the main result of the article.[19]

Proposition 2 *Suppose Assumptions A1–A4 hold. For appropriate $D > 0$ and slopes σ_N, the family of σ_N-VCG mechanisms is robustly asymptotically $D(\frac{1}{N} + \phi_N)$-efficient.*

More precisely, for every slope $\sigma \geqslant 0$ and economy of size $N > 2$ that satisfies A1, A2, A3, and $A4^N$, the σ-VCG mechanism satisfies MC, IR, IC, δ-BS, and ε-Eff, where

$$\delta = N\left[D_1^N\left(\frac{m_o}{N-1} + \phi_N\right) - D_2^N \sigma\right]_+, \quad \varepsilon = D_3 \sigma,$$

18. The second and the third clause in the definition are relevant only when S_i is finite. It follows form strict monotonicity of $q_i^\sigma(s)$ in s_i, established in the proof of Proposition 2, that $s_i(x)$ is a function.

19. The "$[\cdot]_+$" refers to the positive part of a number, $[x]_+ = \max\{0, x\}$.

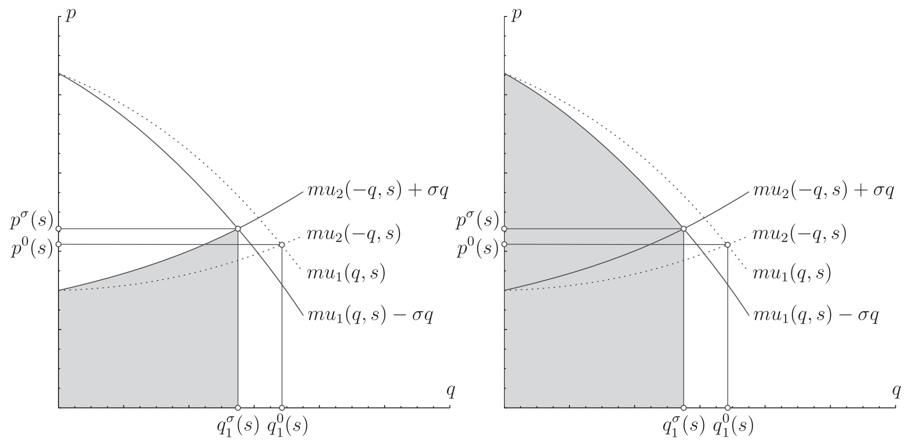

FIGURE 1

σ-VCG allocation and discriminatory transfers in a two-person economy with private values, for $\sigma>0$. Dotted lines are the marginal utility curves, solid lines adjust for the linear per-unit tax. Shaded areas are the discriminatory payments by the buyer (left panel) and proceeds for the seller (right panel).

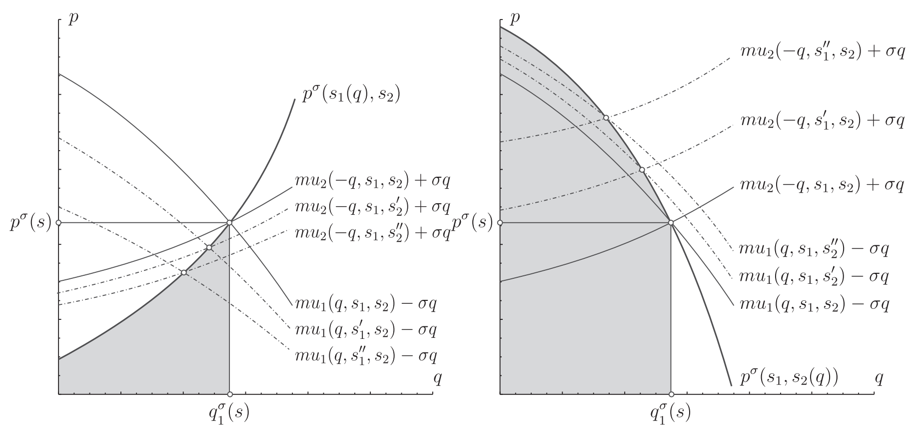

FIGURE 2

σ-VCG allocation and discriminatory transfers in a two-person economy with interdependent values, for $\sigma>0$. Thin lines are the linear tax-adjusted marginal utility curves for the realized type profile (solid), and lower types for the buyer on the left panel and higher types for the seller on the right panel (dotted). Thick solid lines are the residual demand curves, and shaded areas are the discriminatory payments by the buyer (left panel) and proceeds for the seller (right panel).

$$D_1^N=\frac{\overline{q}^2}{2}\frac{\overline{m}_q}{\underline{m}_o-\phi_N},\quad D_2^N=\frac{\overline{q}^2}{2}\left[1-\frac{1}{\underline{m}_o-\phi_N}\left(\frac{\underline{m}_o}{N-1}+\phi_N\right)\right],$$
$$D_3=\frac{\overline{q}}{\underline{m}_q}\left[1+\min\left\{1,\frac{(\overline{m}_q-\underline{m}_q)}{2\underline{m}_q}\right\}\right]. \tag{3.12}$$

Moreover, in every economy of size $N>2$ that satisfies A2, A3, and $A4^N$ only, the σ-VCG mechanism satisfies MC, IR, IC, and BS as long as $\sigma \geqslant \sigma_N$, with

$$\sigma_N = \frac{D_1^N}{D_2^N}\left(\frac{m_o}{N-1}+\phi_N\right). \tag{3.13}$$

The proposition says that for any profile of utility functions, the σ_N-VCG mechanism satisfies all the ex post constraints. Moreover, for a fixed economy size N, it provides an explicit uniform (or "worst case") bound, across all profiles of utility functions, agents, and states, on how much the implemented allocation differs from the optimal one. It follows from Proposition 1 that the bound is of the tight order, and so any mechanism that satisfies the constraints will achieve a proportional worst-case efficiency loss. The result is more general: for smaller slopes, $\sigma<\sigma_N$, σ-VCG mechanisms achieve the tradeoff between budget deficit and inefficiency of the optimal order, as established in Proposition 1.

In case of large economies, as long as Assumption $A4$ of small pairwise interdependence is met, the σ_N-VCG mechanisms are approximately efficient. Moreover, as long as the interdependence parameter ϕ_N is of order smaller than $\sqrt{1/N}$, as in Examples 1 to 3, Lemma 1 implies that the σ_N-VCG mechanisms guarantee vanishing deadweight loss, uniformly across all profiles of utility functions and states:

$$\left|\sum\nolimits_{i=1}^{N} u_i(q_i^{\sigma_N}(s),s)-\sum\nolimits_{i=1}^{N} u_i(q_i^{0}(s),s)\right| \leqslant D_3^2\sigma_N^2\frac{N\overline{m}_q}{2}=O\left(\frac{1}{N}+\phi_N+N\phi_N^2\right).$$

The result also shows that the deficits of the efficient VCG mechanisms are uniformly bounded for every economy size, both in the private value environment ($\phi_N=0$), as well as when the interdependence is of order $\frac{1}{N}$. The required subsidies need not be small; in particular, they are proportional to $\overline{q}^2$.

Let us sketch the intuition behind the results, and justify the constants that determine the magnitudes of the deficit and the efficiency loss. First, incentive compatibility of the mechanisms relies on the logic of pivotal types as in Proposition 1. Specifically, agent i is pivotal for $x'th$ unit of the good precisely when his value for it equals $mu_i\left(x,\left(s_i(x),s_{-i}\right)\right)$. Given the definition of the σ-VCG allocation, this marginal utility equals $p^{\sigma}\left(s_i(x),s_{-i}\right)+\sigma x$. Integrating over such per-unit payments gives rise to the σ-VCG transfers. Monotonicity of the allocation follows from Assumption $A4^N$.

Establishing budget surplus of the σ_N-VCG mechanism is the centrepiece of the result. The first step mirrors the intuition below Proposition 1 to bound the budget deficit of the efficient VCG mechanism. As there, the bound is proportional to the maximal sensitivity of the Walrasian price with respect to the agent's allocation, $\frac{\partial p^0}{\partial q_i}$, and the square of the maximal quantity traded. The maximal sensitivity $p_q^{\max}$ is $\frac{\overline{m}_q}{\underline{m}_o-\phi_N}(\frac{m_o}{N-1}+\phi_N)$, attained in a model with maximally steep demand functions and maximal interdependence (see discussion under Proposition 1). Small pairwise interdependence, $A4$, implies that this effect on price is small in a large economy.

Second, we argue that for the right choice of a slope σ_N, σ_N-VCG mechanisms guarantee elimination of the budget deficit, with efficiency distortion of minimal order. The idea is simple. In the efficient allocation of the model with maximal sensitivity $p_q^{\max}$, buyer i that ends up with $q_i(s)$ units "gets a discount" of $p_q^{\max}\times x$ for the $q_i(s)-x$'th unit he purchases, contributing to the deficit. In a σ-VCG mechanism, with strictly positive slope σ, the sensitivity of price to quantity is proportional to the modified slope of the demand curve $\sigma+\overline{m}_q$, not $\overline{m}_q$ (see (3.9)), resulting in “discounts” of $\frac{\sigma+\overline{m}_q}{\overline{m}_q}p_q^{\max}\times x$ for the $q_i(s)-x$’th unit purchased. On the other hand, setting a quadratic tax $\frac{1}{2}\sigma\times q_i^2(s)$ is equivalent to setting a per-unit tax of $\sigma\times x$ on every inframarginal $q_i(s)-x$’th unit bought. Consequently, when σ_N is picked to equate σ with $\frac{\sigma+\overline{m}_q}{\overline{m}_q}p_q^{\max}$, each trader pays in taxes exactly what he gains via information rents, resulting in a budget surplus with minimal distortions. This argument shows that budget deficit will be eliminated even when the quantities traded are unbounded, as stated in the proposition.

Regarding the efficiency losses, a σ-VCG mechanism distorts marginal utilities from the σ-VCG price by at most $\sigma\overline{q}$. Moreover, we show in the proof that the difference between the Walrasian price and the σ-VCG price is bounded above by $\sigma\overline{q}\min\left\{1,\frac{(\overline{m}_q-\underline{m}_q)}{2\underline{m}_q}\right\}$. For example, when all demand functions have the same slope, the two prices agree, because at the Walrasian price, the distortions imposed by the σ -VCG mechanism on the quantities allocated to the buyers and sellers cancel out. This argument bounds the total wedge between the implemented marginal utilities and the Walrasian price. Multiplying by $\frac{1}{\underline{m}_q}$ yields a bound on the wedge between the implemented and the efficient quantities (by Assumption $A2$), which is precisely the expression for ε in (3.12).

Finally, we note that for any level of permissible deficit, appropriate σ-VCG mechanisms guarantee efficiency distortion that is proportional to, but not exactly equal to the necessary efficiency distortion established in Proposition 1. Specifically, constants C_1^N in Proposition 1 and D_1^N in Proposition 2 capture, roughly, a lower and an upper bound on the deficits of the mechanisms due to “discounts” for the inframarginal units (see above). The discounts are proportional to the square of the maximal quantity traded, which lies within $\frac{N-1}{N}\overline{q}-\varepsilon$ and $\overline{q}$. This gap is reflected in the gap between C_1^N and D_1^N, and it vanishes in large, near-efficient economies. Moreover, the upper bound on the per-player contribution to the budget due to giving up ε efficiency, proportional to C_2 in Proposition 1, is larger than the bound achieved by the σ-VCG mechanisms, proportional to D_2^N/D_3 in Proposition 2. Even in large economies with small pairwise interdependence (ϕ_N small), C_2 is approximately $2\frac{\overline{m}_q}{\underline{m}_q}$ times larger than D_2^N/D_3.

The first reason for the slack is that in the σ-VCG mechanisms, all traders always have their marginal utilities distorted proportionally to the same, conservatively large σ. Fine-tuning the mechanism to the details of the utility function profile can lead to smaller distortion of allocated quantities. Specifically, if some traders in the economy have relatively flat demand curves, they should face smaller distortions in marginal utilities, at least for some quantities. At minimum, implementation of such mechanisms would be non-anonymous, with, say, tax coefficients indexed by agents, as well as agent-specific trade levels (see Section 5). The second reason for the slack is that σ-VCG mechanisms achieve a tighter efficiency objective. They reach the bound ε on the efficiency loss only when the maximal allowed quantity is traded. For smaller quantities, the distortion is proportionally smaller.

Proposition 5 in Section 7 shows formally that those two features of the σ-VCG mechanisms are responsible for the slack in the efficiency distortion. Specifically, when demand functions of all traders have equal slope, and the efficiency loss is defined not as the distance to the efficient allocation but as a fraction of the allocation that this distance makes up, σ-VCG mechanisms strike the right balance between the budget deficit and the efficiency loss, with no slack. Similarly, when budget deficit is eliminated using taxes, which condition only on quantities, but not on agents’ identities or details of utility functions, minimal efficiency distortions require the use of σ_N-VCG mechanisms (Proposition 4 in Section 6).

4. EXAMPLES

In the following, we derive σ-VCG mechanisms for the examples from Section 2. For each example, we also look at information structures with Normal beliefs that can be used to justify the utilities (see Section 2) and compare the σ-VCG mechanisms with the linear Bayesian equilibria in standard double auctions.

Example 4 *Consider the fundamental value model from Example 1. For any σ, the σ-VCG price and allocation are*

$$p^{\sigma}(s) = (\alpha+\beta)\bar{s}, \tag{4.14}$$
$$q_i^{\sigma}(s) = \frac{\alpha}{\mu+\sigma}(s_i - \bar{s}).$$

Using Proposition 2, the slope that guarantees BS equals

$$\sigma_N = \frac{\mu(\alpha+\beta)}{(N-2)\alpha-\beta}, \tag{4.15}$$

which is positive, given A4^N (see Example 1). For $\sigma = \sigma_N$, the mechanism boils down to

$$q_i^{\sigma_N}(s) = \frac{(N-2)\alpha-\beta}{\mu(N-1)}(s_i - \bar{s}), \quad t_i^{\sigma_N}(s) = p^{\sigma_N}(s) \times q_i^{\sigma_N}(s), \tag{4.16}$$

and so satisfies budget balance. This follows from the fact that in this example, $\frac{\partial p^{\sigma_N}(s_i(x), s_{-i})}{\partial x} = \sigma_N$, as can be verified from (4.14).

Consider now a Bayesian framework as in Vives (2011) and Rostek and Weretka (2012). Each agent i has a linear-quadratic utility function $v_i(\theta_i, q_i)$ as in (2.5), and agents believe their values $(\theta_1, ..., \theta_N)$ are jointly normally distributed with mean zero, variances 1, and covariances $\rho \geqslant 0$. Each agent i observes a signal $s_i = \theta_i + \varepsilon_i$, with noise $\varepsilon_i \sim N(0, \zeta)$, $\zeta > 0$, independent of all other variables. One may think of private value $\theta_i = \theta + \theta_i^{id}$ as consisting of a common shock $\theta \sim N(0, \rho)$ and an idiosyncratic shock $\theta_i^{id} \sim N(0, 1-\rho)$.

Applying Proposition 2 from Rostek and Weretka (2012) to this information structure, we verify in Appendix C that

$$\mathbb{E}[v_i(\theta_i, q_i)|s] = \left(\frac{1-\rho}{1-\rho+\zeta} s_i + \frac{N\rho\zeta}{(1-\rho+\zeta)(1+(N-1)\rho)}\bar{s}\right) q_i - \frac{\mu}{2} q_i^2, \tag{4.17}$$

and the linear Bayesian equilibrium in a double auction results in the identical allocation and transfers as the σ_N-VCG mechanism, given by (4.16).

The main takeaway is that in the fundamental value model the σ_N-VCG mechanism—with the lowest slope σ_N still consistent with BS—agrees with, and so can be implemented by, a double auction. Because a demand schedule in a double auction allows an agent to condition his demand on own signal and the equilibrium price, given that the price is privately revealing (*i.e.* carries the same information as the type vector, Rostek and Weretka (2012)), agents are effectively choosing ex post optimal price-quantity pairs. Moreover, in this example, the equilibrium demand reduction and the positive slope σ_N of σ_N-VCG mechanisms have the same effect on allocations and transfers. In particular, it follows that this Bayesian equilibrium remains an equilibrium for any information structure justifying payoffs as in Example 1.

Example 5 *Consider the group model from Example 2, for the simplest case of symmetric agents, with $\alpha_1=\alpha_2$ and $\beta_1=\beta_2$ (see Rostek and Weretka, 2012). For any σ, the σ-VCG allocation and σ-VCG price satisfy*

$$p^{\sigma}(s)=(\alpha+\beta)\bar{s},$$

$$q_i^{\sigma}(s)=\frac{\alpha s_i+\beta\bar{s}^G-(\alpha+\beta)\bar{s}}{\mu+\sigma},\quad \forall i\in G, G=1,2.$$

The slope that guarantees BS equals (Proposition 2)

$$\sigma_N=\frac{\mu(\alpha+\beta)}{(N-2)\alpha},$$

and just as in Example 4, it is easy to verify that the corresponding transfers are $t_i^{\sigma_N}(s)=p^{\sigma_N}(s)\times q_i^{\sigma_N}(s)$, and so satisfy budget balance.

Suppose now agents have utility functions $v_i(\theta_i,q_i)$ as in (2.5) and believe their values $(\theta_1,...,\theta_N)$ are jointly Normally distributed with mean zero, variances 1, and covariances $\rho\geqslant 0$ for the agents only in the same group. Thus, private value $\theta_i=\theta_G+\theta_i^{id}$ consists of a shock $\theta_G\sim N(0,\rho)$, $i\in G$, common to own group, and an idiosyncratic shock $\theta_i^{id}\sim N(0,1-\rho)$. As before, each agent i observes a signal $s_i=\theta_i+\varepsilon_i$, $\varepsilon_i\sim N(0,\zeta)$. Just as in Example 4, it follows that the expected utilities conditional on a profile of signals take the form of utilities in the fundamental value model, for own subgroup: for any $i\in G, G=1,2$,

$$\mathbb{E}[v_i(\theta_i,q_i)|s]=\left(\frac{1-\rho}{1-\rho+\zeta}s_i+\frac{(N/2)\rho\zeta}{(1-\rho+\zeta)(1+(N/2-1)\rho)}\bar{s}^G\right)q_i-\frac{\mu}{2}q_i^2. \tag{4.18}$$

Applying Proposition 2 from Rostek and Weretka (2012) to this information structure, we verify in Appendix C that in the linear Bayesian equilibrium in a double auction, the equilibrium price and allocation are

$$p^{l,N}(s)=\frac{c_s^N}{1-c_p^N}\bar{s},\quad q_i^{l,N}(s)=\frac{c_s^N}{\mu}\left(\frac{\frac{N-2}{N-1}-c_p^N}{1-c_p^N}\right)(s_i-\bar{s}), \tag{4.19}$$

for constants c_s^N, c_p^N that depend on ζ,ρ,N such that

$$\lim_{N\to\infty}c_s^N=\frac{2-\rho}{2-\rho+2\zeta},\quad \lim_{N\to\infty}c_p^N=\frac{2\zeta}{2-\rho+2\zeta}.$$

Allocations and transfers in the two mechanisms now differ. Figure 3 shows ratios of expected utilities in each mechanism, for different correlations of values ρ within each group, noise variances ζ, and economy sizes N. In particular, when the economy grows, expected utilities converge to efficiency only in the case of σ_N-VCG:

$$\lim_{N\to\infty}\mathbb{E}[v_i(\theta_i,q_i^{\sigma_N}(s))|s]=\lim_{N\to\infty}\mathbb{E}\left[q_i^{\sigma_N}(s)\left(\mathbb{E}[\theta_i|s]-\frac{\mu}{2}q_i^{\sigma_N}(s)\right)\right] \tag{4.20}$$

$$=\mathbb{E}\left[q_i^0(s)\left(\mathbb{E}[\theta_i|s]-\frac{\mu}{2}q_i^0(s)\right)\right]=\frac{\mu}{2}\mathbb{E}\left[q_i^0(s)^2\right]=\frac{1}{\mu}\left(\frac{\rho}{4}+\frac{(1-\rho)^2}{2(1-\rho+\zeta)}\right),$$

$$\lim_{N\to\infty}\mathbb{E}\left[v_i(\theta_i,q_i^{l,N}(s))|s\right]=\frac{\mu}{2}\mathbb{E}\left[q_i^{l,\infty}(s)^2\right]=\frac{1}{\mu}\left(\frac{(2-\rho)^2}{4(2-\rho+2\zeta)}\right)<\frac{1}{\mu}\left(\frac{\rho}{4}+\frac{(1-\rho)^2}{2(1-\rho+\zeta)}\right),$$

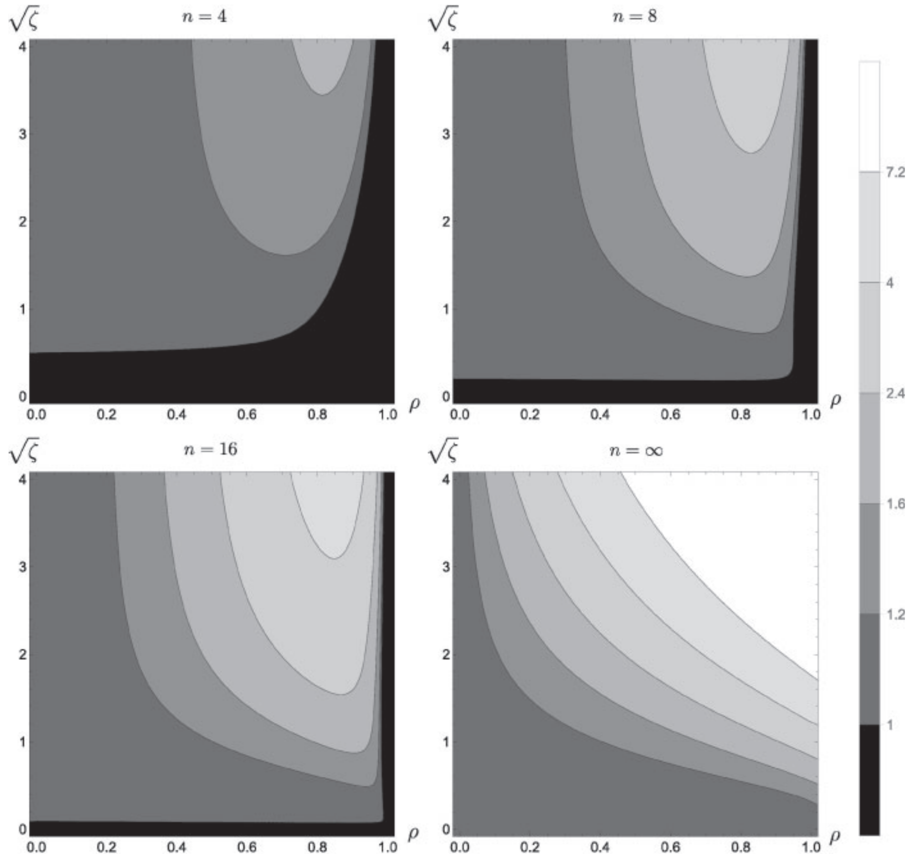

FIGURE 3

Ratios of σ_N-VCG mechanism to double auction expected utilities in a group model, for different correlations within group ρ, signal noise variances ζ, and economy sizes N.

for every $i\in G, G=1,2$, where replacing the order of limits under the expectation may be justified by the Dominated Convergence Theorem, and the fourth equality is justified by $\mathbb{E}[q_i^0(s)(\mathbb{E}[\theta_i|s]-\mu q_i^0(s))]=0$.

When the economy is large, in a σ-VCG mechanism, agents can trade across groups to exploit the difference between the realized group shocks, θ_1 and θ_2, based on the arbitrarily precise estimates $\bar{s}^1$ and $\bar{s}^2$. This trade is captured by the first term in the limiting expected utility in (4.20). The second term represents the benefit of trading away the idiosyncratic part of agent i's value, θ_i^{id}, given the idiosyncratic part of his signal, $\theta_i^{id}+\varepsilon_i$. In the linear Bayesian equilibrium of the double auction, price shifts each agent's own value estimate in step. Consequently, the trade of each agent i is based solely on the information about θ_i provided by his private signal s_i. In particular, large gains from intergroup trade are not realized. For example, large noise ζ of private signals may render trade in a double auction nearly useless, without wiping out the gains from trade in a σ-VCG mechanism (see bottom right panel in Figure 3).

Example 6 *In the fundamental value model with heterogeneous traders from Example 3, for any σ, the σ-VCG allocations and prices satisfy*

$$q_i^\sigma(s) = \frac{\alpha s_i + \beta \overline{s} - p^\sigma(s)}{\mu_G + \sigma}, \quad \forall i \in G, G = S, B,$$
$$p^\sigma(s) = \alpha\left(\gamma_S \overline{s}^S + \gamma_B \overline{s}^B\right) + \beta \overline{s},$$

where $\overline{s}^S$ and $\overline{s}^B$ are average signals in each group and $\gamma_G = \frac{\sigma + \mu_{-G}}{2\sigma + \mu_B + \mu_S}$, $G = S, B$. Compared to the fundamental value model, now big agents trade more aggressively on their information and thus have a larger price impact. Given the allocations and prices, we compute the transfers as in the previous examples (see Appendix C).

Although the same Bayesian model as in Example 4 can be used to justify the utility functions, many more models can. Specifically, suppose agents have utilities $v_i(\theta_i, q_i)$ as in (2.5), and believe that, for $i \in G, G = S, B$,

$$\theta_i = \theta + \theta_G + \theta_i^{id},$$
$$s_i = \theta_i + \varepsilon_G + \varepsilon_i,$$

where all random variables $\theta, \theta_G, \theta_i^{id}, \varepsilon_G, \varepsilon_i^{id}$, for $G = S, B$ and $i \leqslant N$, are independently Normally distributed with variances $\sigma_\theta^2, \sigma_{\theta_G}^2, \sigma_{\theta_{id}}^2, \sigma_{\varepsilon_G}^2, \sigma_{\varepsilon_{id}}^2$. In other words, everyone's value experiences a common shock θ, either of the two group shocks θ_S, θ_B, and an idiosyncratic shock θ_i^{id}. Likewise, noise consists of a group noise component $\varepsilon_G, G = S, B$, and idiosyncratic noise ε_i. Intuitively, agents in the same group care about similar aspects of the asset and also observe signals from similar sources, giving rise to positively correlated noises.[20]

With no group value and noise shocks, we recover the model from Example 4 as a special case. However, for fixed parameters α and β in the utility function, a continuum of Bayesian models from this class exist, with variances solving the system of linear equations (see Appendix C):

$$\sigma_{\theta_{id}}^2 = \frac{\alpha \sigma_{\varepsilon_{id}}^2}{1-\alpha}, \quad \sigma_{\theta_G}^2 = \frac{\alpha \sigma_{\varepsilon_G}^2}{1-\alpha}, \quad \sigma_\theta^2 = \frac{\beta\left(2\sigma_{\varepsilon_{id}}^2 + N\sigma_{\varepsilon_G}^2\right)}{2N(1-\alpha)(1-\alpha-\beta)}. \tag{4.21}$$

For example, increasing common group noise $\sigma_{\varepsilon_G}^2$, which decreases the attractiveness of signals from own group, is compensated by the increase in correlation $\sigma_{\theta_G}^2$ of values within a group.

This continuum of models gives rise to a continuum of different linear Bayesian equilibria in double auctions. Moreover, for a fixed vector of parameters that solve (4.21), generically an analytical characterization of the equilibrium is unavailable (see Appendix C). However, we may argue indirectly as follows. The same linear equilibria result in the same linear market clearing price $p^l(s) = \gamma_S \overline{s}^S + \gamma_B \overline{s}^B$, for appropriate γ_S, γ_B, and the same price impact $\frac{\partial p^l}{\partial q_i} = d_i \geqslant 0$, for any agent i. Agent i's demand $q_i^l(s_i, p)$, given signal s_i and price p^l, satisfies

$$q_i^l(s_i, p) = \frac{E\left[\theta_i | s_i, p^l\right] - p^l}{\mu_G + d_i}, \tag{4.22}$$

20. Other Bayesian models are available. We assumed that agents that share a group value shock and have positively correlated noises also share the degree of risk aversion. The two splits could disagree, or more groups with correlated values and signals could exist. Likewise, variances of either group value shocks or idiosyncratic value components could differ (see Section 7).

for $i\in G, G=S,B$. As we show in the Appendix C, the expectation function $E\left[\theta_i|s_i,p^l\right]$ can agree for at most countably many models.

Despite the differences in risk aversion among the agents, in the model with uncorrelated noises ($\sigma^2_{\varepsilon_G}=\sigma^2_{\theta_G}=0$), expected utilities converge to the efficient ones (Manzano and Vives, 2016, see also Hellwig, 1980). However, this model is the only one featuring asymptotic efficiency. It follows from equation (4.22) that with vanishing price impacts, as long as $\mu_B\neq\mu_S$, the market clearing price $p^l(s)=\gamma_S\bar{s}^S+\gamma_B\bar{s}^B$ may not assign equal weights to the two group averages, $\gamma_S\neq\gamma_B$. As long as noises are positively correlated within a group ($\sigma^2_{\varepsilon_G},\sigma^2_{\theta_G}>0$), neither $\bar{s}^S$ nor $\bar{s}^B$, and thus also neither the price, converges in probability to the average of all signals $\bar{s}$. It follows that $E\left[\theta_i|s_i,p^l\right]\neq E[\theta_i|s_i,\bar{s}]$, and so, from (4.22), allocations do not converge to the efficient ones.

The example shows how a continuum of Bayesian models underlying a fixed utility profile may (i) feature a continuum of Bayesian equilibria in double auctions that are sensitive to the details of the underlying information structure, and that (ii) (generically) lack analytical characterization, and (iii) (generically) do not yield asymptotic expected efficiency. All those features contrast with uniqueness, simplicity, and asymptotic efficiency of the σ_N-VCG mechanisms.

5. IMPLEMENTATION

σ-VCG mechanisms are direct mechanisms: agents report their types to the mechanism designer, who enforces the reallocation of the good and the payments. Although robustness strips him of the responsibility to know the information structure, he must know the utility functions. In this section, we present a way to implement σ-VCG mechanisms that does not require any such knowledge.

We make the following two assumptions (see Section 7 for discussion):

(INF) $\quad S_i=\mathbb{R},\ \forall i.$
(INJ) $\quad \inf_{q,s}\det\left[\,mu_{i,j}(q_i,s)\,\right]_{i,j\leqslant N}>0.$

For the linear-quadratic utility functions, the INJ needs to be checked only at a fixed pair (q,s), and is easily verified in each of the examples from Section 3.

Fix slope $\sigma\geqslant 0$ and consider the following auction format.

Definition 3 *(σ-Discriminatory Conditional Double Auction, σ-DCDA)*

Actions. *Each agent i chooses a continuous* conditional inverse demand correspondence,

$$d_i:\mathbb{R}^{N-1}\twoheadrightarrow\mathbb{R},$$

where $d_i(q_{-i})$ is the set of marginal prices at which i is willing to clear the market and purchase $\sum_{j\neq i}q_j$, when every other agent $j\neq i$ is allocated q_j.

Allocation. *The mechanism finds the price-allocation pair (p,q) that clears the market:*

$$p\in d_i(q_{-i}),\ \ \forall i. \tag{5.23}$$

Transfers. *A continuous function $p_i:\mathbb{R}\to\mathbb{R}$ is a* residual demand curve *for i, $i\leqslant N$, if for every q_i, q_{-i} exists such that the price-allocation pair $(p_i(q_i),q_i,q_{-i})$ clears the market. Transfers by i are defined as*

$$t_i=\int_0^{q_i}p_i(x)dx+\frac{\sigma}{2}q_i^2,$$

for the residual demand curve p_i.

If, for the submitted conditional inverse demands, no unique residual demand curve exists for every $i \leqslant N$, and so there is no unique market clearing pair (p,q), then the allocation and transfers are zero.

The first step in running a σ-DCDA, which is finding the price-allocation vector, requires solving a system of N inclusions (5.23) in $N+1$ unknowns (p,q), together with the market clearing condition. Aside from the added difficulty in allowing for set-valued demands discussed below, this step generalizes the fixed-point problem of finding the market clearing price in a double auction. It boils down to it when inverse demands condition only on the total quantity to be cleared.

Again, much like in a double auction, reports by all agents other than i determine the residual demand curve that i is facing. It is the menu of (marginal) price-quantity pairs (p,q) that i is choosing from, which is pinned down by the market clearing and $N-1$ demands submitted by other agents. One term in the transfers is a tax, quadratic in the quantity traded, paid by every agent. The other term is the discriminatory payment: an integral over all the inframarginal units under the residual demand curve that i is facing.[21]

We note that the discriminatory nature of payments refers to the trader paying different prices for each unit he trades. Different discriminatory payments are used in practice, such as when a trader pays the area under his own submitted demand curve. Such payments are analogous to "paying-own-bid" in the first-price auction. Traders paying the area under the residual curve they face, in our auction, is analogous to the payments in the second-price auction.

Consider the following bids in the auction. For any $\sigma \geqslant 0$, a σ-VCG *conditional inverse demand* strategy in a σ-DCDA for agent $i \leqslant N$ is defined as

$$d_i(s_i)(q_{-i}) = \left\{ mu_i(-\sum_{j\neq i} q_j, s_i, s_{-i}) + \sigma \sum_{j\neq i} q_j \,\middle|\, q^{\sigma}_{-i}(s_i, s_{-i}) = q_{-i} \right\}.$$

In the case of $\sigma = 0$, the VCG conditional inverse demands are reports of own marginal utilities, evaluated at the efficient allocation. In the case of private values, the conditioning is vacuous, and the agents are simply reporting their marginal utilities $mu_i(q_i, s_i)$, for all $q_i \in \mathbb{R}$.[22] The bids are analogous to the bids in the second-price auction, in the standard auction setting. When the values are interdependent, conditioning on the efficient allocation is a way to provide agents with additional information about the types of other agents. Such information is necessary for the efficient trade. Finally, in the case of strictly positive slope $\sigma > 0$, agents report their modified, tax-adjusted marginal utilities.

The following example solves for the σ-VCG conditional inverse demands in case of linear utilities.

Example 7 *In case of linear quadratic utilities, as in the previous examples, σ-VCG allocations satisfy*

$$q_i^{\sigma}(s) = \frac{\sum_j w_{ij} s_j - p^{\sigma}(s)}{\mu_i + \sigma},$$

21. Given few restrictions on bids, they need not determine unique allocation or transfers, in which case the economy reverts to autarky. This proviso is standard in double auctions when multiple market clearing prices exist. Alternatives such as picking an allocation or residual demand curves according to some pre-specified order would do just as well. Of course, we verify in the proof of the following Proposition that the multiplicity does not arise in equilibrium.

22. Strictly speaking, for q_i that do not arise in any efficient allocation consistent with s_i, the agent is reporting 0. The difference does not play any role.

where $w_{ij} = mu_{i,j}(0,0)$*, which implies*

$$p^{\sigma}(s) \cdot \sum_j \widetilde{w}_{ij} = \left(s_i - \sum_j \widetilde{w}_{ij}(\mu_j + \sigma) q_j^{\sigma}(s) \right) \tag{5.24}$$

$$= \left(s_i - \sum_{j \neq i} (\widetilde{w}_{ij}(\mu_j + \sigma) - \widetilde{w}_{ii}(\mu_i + \sigma)) q_j^{\sigma}(s) \right),$$

where $\left[\widetilde{w}_{ij}\right]_{i,j \leqslant N}$ *is the inverse of the matrix* $\left[w_{ij}\right]_{i,j \leqslant N}$.

In the fundamental value model, matrix inversion yields σ*-VCG conditional inverse demand functions*

$$d_i(s_i)(q_{-i}) = (\alpha + \beta) \left[s_i + \frac{\mu + \sigma}{\alpha} \sum_{j \neq i} q_j(s) \right],$$

which depend only on the total quantity an agent must clear.

By contrast, in the group model from Example 2, in the symmetric case when $\alpha_1 = \alpha_2$ *and* $\beta_1 = \beta_2$*,* σ*-VCG conditional inverse demand functions are*

$$d_i(s_i)(q_{-i}) = (\alpha + \beta) s_i + \frac{\mu + \sigma}{\alpha} \left[\left(\alpha + \beta \frac{N-2}{N} \right) \sum_{j \neq i} q_j + \beta \overline{q}^{\text{own}} \right].$$

The marginal price at which an agent is willing to clear the market is increasing in own signal and in the quantity $\sum_{j \neq i} q_j$ *that he must clear, due to the decreasing marginal utility of absorbing more units. Unlike in a standard inverse demand function, however, each agent conditions his price also on the average trade by the agents in his group. For a fixed own type and total quantity to be cleared, higher demand by own group indicates higher own group shock and so value of the good to the agent, thus increasing the price.*

Similarly, in the fundamental value model with heterogeneous traders, σ*-VCG conditional inverse demand by a big agent i is (and similarly for small agents):*

$$d_i(s_i)(q_{-i}) = (\alpha + \beta) \left[s_i + \frac{\mu + \sigma}{\alpha} \sum_{j \neq i} q_j(s) \right] + \frac{\beta(\mu_S - \mu_B)}{2\alpha} \overline{q}^S.$$

Although small agents have equally precise information as big agents, they trade less aggressively. Thus, keeping the total demand by all other agents fixed, high demand by small agents reveals high value of the good and so pushes up the price demanded to clear the market.

The following is the second main result of the article. Recall that an ex post equilibrium is the game theoretic counterpart of ex post incentive compatibility. It means that each agent's strategy is optimal at every type profile of other agents, and so irrespectively of the underlying information structure (see Bergemann and Morris, 2005).

Proposition 3 *Fix an economy of size* $N > 2$ *that satisfies assumptions INF, INJ, A2, and* $A4^N$*. Fix* $\sigma \geqslant 0$ *and consider a game induced by the* σ*-DCDA. The profile of* σ*-VCG conditional inverse demand strategies constitutes an ex post equilibrium that results in transfers and allocation as in the* σ*-VCG mechanism.*

The idea behind the result is as follows. Assumptions INF and INJ are sufficient for the function from signal to marginal utility profiles, evaluated at any allocation, to be globally invertible. Crucially, given the definition of σ-VCG, this invertibility implies that the function from signal profiles to σ-VCG allocation-price pairs is invertible as well, for any $\sigma \geqslant 0$. But this, intuitively, establishes that the informational content of either is the same. More precisely, when each agent i reports the σ-VCG conditional inverse demand $d_i(s_i)$, such action is equivalent to reporting the set of σ-VCG allocation-price pairs consistent with own signal s_i. Injectivity implies that the intersection of the reports across all the agents is the unique σ-VCG price-allocation $(p^\sigma(s), q^\sigma(s))$. Uniqueness of residual demand curves follows analogously, and ex post optimality follows from IC of the σ-VCG mechanism.

The conditional inverse demands in our implementation are related to the conditional bids in Dasgupta and Maskin (2000) and Eso and Maskin (2002), in an auction setting. In each paper, they serve the same purpose of allowing the expression of own demand as a function of types of other agents, and a fixed point has to be solved to determine the allocation. In Dasgupta and Maskin (2000) as well as Eso and Maskin (2002), bids are conditioned on vectors of opponents' utility *functions,* where a utility function assigns utility to each allocation a trader could get. In our setting, such vectors of functions would be elements in $\left(\mathbb{R}^{\mathbb{R}}\right)^{N-1}$.[23] Our inverse demands are conditioned on vectors of opponents' allocations, and so elements of $\mathbb{R}^{N-1}$. We show that the allocation vectors carry enough information to condition on. The bids are thus simpler; formally, the set of all the available bids has a smaller dimension.[24]

The extent to which σ-VCG conditional inverse demands are complicated reflects the relatively unrestricted nature of the primitives in our framework. On the other hand, as Example 7 shows, when utilities are simple and depend only on the low-dimensional statistic of the types of others, the inverse demands are also simple. This is not a coincidence:

Lemma 2 *Fix n, $0 \leqslant n < N$, and suppose that for every i, functions $w_i : S_{-i} \to \mathbb{R}^n$ and $\widetilde{u}_i : \mathbb{R}^{n+2} \to \mathbb{R}$ exist such that*

$$u_i(\cdot, s_i, s_{-i}) = \widetilde{u}_i(\cdot, s_i, w_i(s_{-i})).$$

Then a function $\widetilde{w}_i : \mathbb{R}^N \to \mathbb{R}^n$ exists such that σ-VCG conditional inverse demand satisfies

$$d_i(s_i)(q_{-i}) = \left\{ x | x = m\widetilde{u}_i(-\sum\nolimits_{j \neq i} q_j, s_i, \widetilde{w}_i(q_{-i}, x)) + \sigma \sum\nolimits_{j \neq i} q_j \right\}, \ \forall i. \tag{5.25}$$

If $d_i(s_i)$ is a function, then there is $\widetilde{w}_i : \mathbb{R}^{N-1} \to \mathbb{R}^n$ such that

$$d_i(s_i)(q_{-i}) = m\widetilde{u}_i(-\sum\nolimits_{j \neq i} q_j, s_i, \widetilde{w}_i(q_{-i})) + \sigma \sum\nolimits_{j \neq i} q_j, \ \forall i. \tag{5.26}$$

Another potential complication is that a σ-VCG conditional inverse demand may be a correspondence. Intuitively, with a lot of interdependence, high shocks of others can move the marginal utilities of all the agents in step, not affecting the allocation. Consequently, an agent may report several marginal utilities consistent with a given allocation. This phenomenon is familiar from the rational expectations or the double auction literature. Too much interdependence may imply upward-sloping demand, resulting in the nonexistence of equilibrium (*e.g.* Kyle, 1989).

23. Dasgupta and Maskin (2000) and Eso and Maskin (2002) consider discrete, heterogeneous goods, and so in those papers, an allocation is a partition of available goods, not a quantity vector.

24. The set of available bids in Dasgupta and Maskin (2000) and Eso and Maskin (2002), or what they condition on, can be reduced to only those that occur in equilibrium. The number of equilibrium bids is determined by the set of traders' signals. From the implementation perspective, however, this reduction requires the designer's knowledge of traders' utility functions, which is precisely the kind of knowledge that the implementation is supposed to do away with.

The following lemma shows a sufficient condition (necessary for linear utilities) for those issues not to arise in our setting.

Consider the following assumption:

$$\text{(FUN)} \qquad \text{The sum of every row in } \left[mu_{i,j}(q_i,s)\right]^{-1}_{i,j\leqslant N} \text{ is non-zero}, \ \ \forall q,s.$$

Lemma 3 *Assumption FUN is sufficient for σ-VCG inverse demands to be functions. In case of linear utilities, it is also necessary.*

Below, we present two sufficient conditions for FUN. The first requires, roughly, that not too much interdependence is present in preferences. Surprisingly, the second sufficient condition is *equicommonality* (Rostek and Weretka, 2012), satisfied when the average level of interdependence across the agents is constant, no matter its level. It is satisfied in all the examples considered in this article. In the Bayesian framework, with Normal beliefs and equal quadratic coefficients, the condition is sufficient for the existence of a symmetric linear equilibrium.

Lemma 4 *Either of the following conditions is sufficient for FUN. For every i,j,q_i,q_j,s,*

$$(i) \qquad mu_{i,i}(q_i,s)\geqslant 2N\cdot mu_{j,i}(q_j,s),$$

$$(ii) \qquad \sum\nolimits_k mu_{i,k}(q_i,s)=\sum\nolimits_k mu_{j,k}(q_j,s).$$

6. OTHER MECHANISMS

One way to interpret a σ-VCG mechanism is as follows. Agents are charged with taxes that are quadratic in the quantity they trade. Then, assuming taxes are internalized by the agents and thus their utilities are properly adjusted, σ-VCG mechanisms are the (generalized) VCG mechanisms that implement the efficient allocation. More precisely, fix a *tax* $\tau:\mathbb{R}_+\to\mathbb{R}_+$, which is twice continuously differentiable, increasing, and convex. A mechanism $\{(q(s),t(s))\}_{s\in S}$ is a *VCG mechanism with taxes* τ (τ-VCG) if

$$q_i(s)=\widetilde{q}_i(s),$$
$$t_i(s)=\widetilde{t}_i(s)+\tau(|\widetilde{q}_i(s)|),$$

where $\{(\widetilde{q}(s),\widetilde{t}(s))\}_{s\in S}$ is an IR, IC, MC mechanism implementing the efficient allocation for utility functions $\widetilde{u}_i$, $i\leqslant N$,

$$\widetilde{u}_i(q_i,s)=u_i(q_i,s)-\tau(|q_i|).$$

A VCG mechanism with taxes τ satisfies IR, IC, and MC for every choice of tax function τ. Taxes allow to trade off efficiency and budget surplus: trivial taxes result in the efficient VCG mechanism and budget deficits. Larger taxes distort efficiency of the allocation, relative to the underlying utilities, but mitigate the budget deficit. The σ-VCG mechanisms strike an optimal balance between efficiency and budget surplus (Propositions 1 and 2), up to a multiplicative constant. In this section, we show that they are, roughly, uniquely optimal in this sense.

Consider the following two examples.

Example 8 ***(Linear tax: bid-ask spread)*** *Fix $\delta\geqslant 0$ and consider a linear tax $\tau(q)=\delta|q|$. For any type profile s, the allocation $q(s)$ of a VCG mechanism with such taxes is determined jointly* *with bid-ask prices* $p_b(s), p_a(s)$ *such that* $p_a(s) = p_b(s) + 2\delta$, *markets clear, and*

$$mu_i(q_i(s), s) = p_a(s), \ \ \forall i \text{ such that } q_i(s) > 0, \tag{6.27}$$
$$mu_i(q_i(s), s) = p_b(s), \ \ \forall i \text{ such that } q_i(s) < 0.$$

Transfers $t(s)$ *that guarantee incentive compatibility and individual rationality are*

$$t_i(s) = T_i(s_{-i}) + \begin{cases} \int_0^{q_i(s)} p_a\left(s_i(x), s_{-i}\right) dx \text{ if } q_i(s) > 0, \\ \int_0^{q_i(s)} p_b\left(s_i(x), s_{-i}\right) dx \text{ if } q_i(s) < 0, \end{cases}$$

and $t_i(s) = T_i(s_{-i})$ *if* $q_i(s) = 0$, *where* $s_i(x)$ *is defined as in (3.11) and* $T_i(s_{-i}) \leqslant 0$.

In other words, the allocation is as if the market determined the market clearing bid-ask price pair with a fixed spread of 2δ, and each agent received all the relevant information and was free to trade at those prices. The discriminatory Vickrey transfers are then chosen to guarantee incentive compatibility and individual rationality.

Example 9 ***(Entry-fee tax)*** *Fix* $\phi \geqslant 0$ *and consider the entry-fee tax* $\tau(q) = \phi \times \mathbf{1}_{q \neq 0}$. *For any type profile* s, *the allocation* $q(s)$ *of the VCG mechanism with such taxes is determined jointly with a price* $p(s)$ *such that markets clear and*

$$mu_i(q_i(s), s) = p(s), \ \ \forall i \text{ such that } q_i(s) \neq 0, \tag{6.28}$$
$$q_i(s) \neq 0 \text{ iff } \int_0^{q_i(s)} mu_i(x, s) dx - p(s) q_i(s) \geqslant \phi.$$

The transfers $t(s)$ *that guarantee incentive compatibility are*

$$t_i(s) = T_i(s_{-i}) + \phi + \int_0^{q_i(s)} p\left(s_i(x), s_{-i}\right) dx, \ \ \text{if } q_i(s) \neq 0,$$

and $t_i(s) = T_i(s_{-i})$ *if* $q_i(s) = 0$, *where the* s_i *function is defined in (3.11) and* $T_i(s_{-i}) \leqslant 0$.

Intuitively, now the allocation is as if the market determined the market clearing price, each agent received all the relevant information and then could decide whether to pay a fixed entry fee ϕ in order to trade at this price.

Proposition 4 *Suppose Assumptions A1–A4 hold. If* τ_N*-VCG mechanisms are robustly asymptotically* $o\left(\sqrt{\frac{1}{N} + \phi_N}\right)$*-efficient, then for every* $q \in (0, \overline{q})$,

$$\lim_{N \to \infty} \left\{ \left(\frac{\underline{m}_o}{N-1} + \phi_N \right)^{-1} \int_0^q \int_0^x \tau_N''(s) ds dx - \frac{\overline{m}_q}{(\underline{m}_o - \phi_N)} \frac{q^2}{2} \right\} \geqslant 0. \tag{6.29}$$

The rate $o\left(\sqrt{\frac{1}{N} + \phi_N}\right)$ of convergence to efficiency that is required in the proposition is strictly slower than that achieved by the σ_N-VCG mechanisms (Proposition 2). The result says that if some VCG mechanisms with taxes converge to efficiency at this rate or faster, then, for large N, taxes must be at least quadratic in quantity. Moreover, the minimal slope that is required for budget surplus—when condition (6.29) binds at every quantity q—converges to the maximal sensitivity $p_q^{\max} = \frac{\overline{m}_q}{\underline{m}_o - \phi_N}(\frac{m_o}{N-1} + \phi_N)$ of the Walrasian price with respect to quantity.[25] Consequently, it converges to the minimal slope σ_N required from the σ-VCG mechanisms.[26]

The proposition thus establishes that any VCG mechanisms with taxes that are robustly asymptotically efficient must result in strictly larger taxes, and thus larger efficiency distortion than σ_N-VCG, for large N. In particular, VCG mechanisms with entry fees or bid-ask spreads perform strictly worse. Below, we argue why those two types of taxes may not patch the budget deficit in a near-optimal way.

When many agents have average signals and are not willing to pay the fee, or their marginal utilities for the first unit fall within the spread, the few trading ones have a large price impact. For example, a small fixed fee of ϕ that a buyer pays results in a steep decrease in prices for his last inframarginal units, before other agents enter the market. Thus, the fee is dwarfed by the decrease in prices for all but the last inframarginal units, and so it may not cover the deficit. Similarly, a spread (or a per-unit fee) of δ may decrease the price for all but the last inframarginal units by δ, which is compounded with the decrease in prices at signals when most agents trade. Although in this case, the budget can be balanced, it requires a large fee and thus a large distortion. Quadratic taxes, which never exclude any traders but only scale down quantities traded, guarantee that the price impact may not be large.

7. DISCUSSION

We would like to discuss and interpret now some of the assumptions behind the results, and point to the possible extensions.

Weaker assumptions. Assumptions on the utilities can be weakened to fit specific applications: (i) Assumptions $A2$ and $A3$ can be qualified to hold only when $mu_i(q_i,s) \in [\min_{i\leqslant N} mu_i(0,s), \max_{i\leqslant N} mu_i(0,s)]$. In particular, this relaxation allows environments with nonnegative marginal utilities at any allocation. (ii) With large economies, when σ-VCG allocations are nearly efficient, Assumptions $A4$ and $A4^N$ can be weakened to hold only locally, at allocations q_{-j} such that all $mu_i(q_i,s)$, $i \neq j$ are close. (iii) When in Assumption $A4^N$, the average interdependence $\frac{1}{N-1}\sum_{i\neq j} mu_{i,j}(q_i,s)$ is replaced by $\max_{i\neq j} mu_{i,j}(q_i,s)$, the scaling constant $\frac{\underline{m}_q}{\overline{m}_q}$ can be dropped.

Multidimensional signals. Without additional restrictions, our results cannot be extended to environments with multidimensional signals. Jehiel et al. (2006) have shown that with multidimensional signals and interdependence, typically no nontrivial allocation can be ex post incentive compatible. Nevertheless, in several cases, such analysis is not doomed. For example, results in Eso and Maskin (2002) and Jehiel et al. (2008) suggest that our analysis may be extended to the case of multidimensional signals as long as marginal utilities are linear functions of signals (see also Bikhchandani, 2006). In case of private value environments, the impossibility has no bite, and results in Bikhchandani et al. (2006) suggest the right notion of monotonicity. Detailed analysis of each extension is beyond the scope of this article.

Tightness. Efficiency losses in Proposition 2 are tight in that no other class of mechanisms converges to efficiency at a faster rate as the economy grows (Proposition 1). Although a tight rate of convergence is a standard optimality criterion in the asymptotic worst-case analysis, we may strengthen the result as follows. Assume that the marginal utilities decrease linearly in quantity,

25. More precisely, it is the ratio of the slope and $p_q^{\max}$ that converges to one,

26. Although σ_N equals $\frac{\sigma_N + \overline{m}_q}{\overline{m}_q} p_q^{\max}$, as discussed below Proposition 2, its ratio over $p_q^{\max}$ converges to 1, under $A4$.

$$(A2') \qquad mu_{i,q}(q_i,s) = -\overline{m}_q < 0, \;\; \forall i, q_i, s.$$

and consider the following alternative measure of ex post efficiency loss,

$$\varepsilon\text{-}q\text{\_Eff}) \qquad \left| q_i(s) - q_i^0(s) \right| \leqslant \varepsilon |q_i(s)|, \;\; \forall s.$$

We define robust asymptotic f_N-q_efficient of a family of mechanisms in an analogous way to robust f_N-efficiency.

Proposition 5 *Suppose Assumptions A1, A2′, A3, and A4 hold. σ_N-VCG mechanisms are robustly asymptotically $\frac{\sigma_N}{\overline{m}_q}$-q_efficient. Moreover, for any $\gamma < 1$, no robustly asymptotically $\gamma\frac{\sigma_N}{\overline{m}_q}$-q_efficient family of mechanisms exist.*

Discrete allocations. Our results extend to environments with discrete allocations, as long as the units traded are small. The discrete-allocation analogues of Propositions 1 and 2, under suitably modified assumptions, are established in Appendix G.

The problem with discrete allocations, when all traders' utilities may lie on a fixed grid with step Δ, is as follows. When a buyer reports a lower signal and decreases his demand, the price might have to drop by the grid step Δ before any of the other traders picks up his demand to clear the market, no matter how big the economy is. In a sense, a buyer gets both a linear discount, as discussed below Propositions 1 and 2, as well as a fixed discount of size Δ for each inframarginal unit he trades.

Specifically, in Proposition 6, we establish a lower bound on the efficiency distortion, which has an additional term of order Δ, for every economy size. On the other hand, following logic similar to that in Proposition 2, in Proposition 7, we show that extending σ-VCG mechanisms with a constant per-unit tax proportional to Δ eliminates the budget deficit with distortions of minimal order.

Near-incentive compatibility. Rather than construct mechanisms that satisfy all the constraints and give up some efficiency (or generate a small budget deficit), one might want to relax the constraints of incentive compatibility.[27] We claim that this approach is problematic, in our exchange setting with ex post constraints.

Consider a direct mechanism that naively implements the Walrasian auction, ignoring incentive problems. Traders report their types, which are taken at a face value, and the efficient allocation is implemented, with traders paying the uniform per-unit Walrasian price. One of the crucial steps in the proof of Proposition 2 establishes that under our assumptions, each trader has a small effect on the Walrasian price, in a large economy ($\frac{\partial p^0}{\partial q_i}$ is small). It implies that truthful reporting of own type is near optimal in the naive Walrasian auction: misreporting distorts quantity, and can have at most a negligible beneficial effect on the per-unit price, of order $\frac{\partial p^0}{\partial q_i} \times \overline{q}$.

Although the mechanism allows for a near-incentive compatible strategies that result in full efficiency, it does not follow that it has a "nearby" non-truthful profile of strategies that is an ex post equilibrium, and sacrifices little efficiency. Indeed, the mechanism does not have any ex post equilibrium, even for the relatively simple economies with private values, say, $u_i(q_i,s) = s_i q_i - q_i^2$.

27. Relaxing individual rationality and budget balance is near analogous: levying fixed payments (subsidies) on all traders mitigates the deficit but exacerbates individual rationality (or vice versa).

The reason is that a trader's optimal report depends on the reports of others.[28] An agent wants to scale down own trade, which requires the knowledge of the baseline quantities traded.

As a consequence, optimal strategies in the naive Walrasian auction must depend on agents' belief hierarchies, and it is not difficult to find hierarchies that result in a very inefficient allocation. Intuitively, to have an ex post equilibrium, a mechanism must use finer instruments, which allow the conditioning of $i's$ allocation and transfer on the type vector of others in a precise way. This conditioning is a central feature of our mechanisms, made explicit in the implementation.

Signals as types. One source of interdependence in preferences is informational, when agent's type corresponds to a signal informative of each agent's value. In a Bayesian model, interdependencies in expected utilities, conditional on a profile of signals, arise naturally when either signals or values are correlated (see Section 2). This informational motivation of the interdependence does not seem to square with the prior-free approach of this article. We note that, despite this objection, one may still view σ-VCG mechanisms as particularly simple Bayesian mechanisms that attain asymptotic efficiency.

However, the problem is only apparent. Ex post utilities and a fixed distribution of correlated signals and values indeed pin down expected utilities with interdependencies, but the reverse is not true. Because many distributions may give rise to one expected utility function, knowing only the latter assumes strictly less than the Bayesian framework. The following example makes the point stark. Suppose agent i has a linear ex post utility function $v_i(q_i,\theta_i)$ as in (2.5), and fix a linear expected utility function

$$u_i(q_i,s)=q_i\sum w_{ij}s_j-\frac{\mu}{2}q_i^2,$$

for some $\mu, w_{ij}, j\leqslant N$. Prior-free analysis requires knowledge of u_i only, whereas a Bayesian game complements u_i with a fixed distribution $\delta_S\in\Delta(S)$. We may assume δ_S is Normal.

Lemma 5 *For any Normal distribution $\delta_S\in\Delta(S)$, a Normal distribution $\delta_{\Theta_i\times S}\in\Delta(\Theta_i\times S)$ exists such that*

$$u_i(q_i,s)=\mathbb{E}_{\delta_{\Theta_i\times S}}[v_i(q_i,\theta_i)|s].$$

The proof is the statement of the Projection Theorem for Normal distributions read as a system of N linear equations in N variables, $Cov(s,\theta_i)$. In other words, expected utilities put no restrictions on the distribution of signals δ_S.

Equilibrium uniqueness. Ex post implementation implies that, in every information structure, the game induced by a σ-VCG mechanism has a truthtelling equilibrium. In the paper, we are not requiring that this equilibrium is unique. The following is a direct consequence of Theorem 1 in Bergemann and Morris (2009). In case of linear utilities, with $mu_{i,i}(q_i,s)$ normalized to 1, $i\leqslant N$, truthtelling is the unique equilibrium of a σ-VCG mechanism in every information structure precisely when the matrix

$$\begin{bmatrix} 0 & |w_{1,2}| & \dots & |w_{1,N}| \\ |w_{1,2}| & 0 & & \\ \dots & & \dots & \\ |w_{1,N}| & & & 0 \end{bmatrix}$$

28. It is easy to verify that the ex post best response of trader i with type s_i to a truthful report of others is $\widetilde{s}_i=\frac{N^2}{N^2-1}\left(\frac{N-1}{N}s_i+\frac{1}{N}\overline{s}_{-i}\right)$, where $\overline{s}_{-i}$ is the average report of other traders.

with $w_{ij}=mu_{i,j}(q_i,s)$, has the largest eigenvalue $\lambda<1$. Moreover, the condition is also necessary for the equilibrium uniqueness under *any* mechanism (Bergemann and Morris, 2009, Section 3).[29] In other words, whether ex post near-efficient implementation can be strengthened to guarantee equilibrium uniqueness in every information structure is not a mechanism design problem, and the answer depends solely on the properties of an exchange economy.

Conditions for implementation. Assumptions INF and INJ in Section 5 are sufficient for the function from the type to the marginal utility profiles (at any allocation) to be globally invertible.[30] Given the definition of σ-VCG mechanisms, it guarantees that the mapping from type profiles to endogenous σ-VCG allocation and price profiles is globally invertible as well. The conditions help achieve a sensible implementation, but are not necessary. If they are violated, a profile of reported σ-VCG conditional inverse demands may result in multiple profiles of market clearing price-allocation pairs and transfers. In such an event, instead of shutting down trade, a mechanism may allow another stage with "all-out" communication: have each agent report his type, and then agents pick the allocation and transfer profile, with the unanimous choice implemented. Although such communication requires a daunting amount of coordination (see Dasgupta and Maskin, 2000), no alternative seems to exist, if a simpler sufficient statistic for the information carried by the types is unavailable.

Appendix A: Proof of Proposition 1

Fix N even, and consider the following utility functions:

$$u_i(q_i,t_i,s)=\left(\left(\underline{m}_o-\phi_N\right)s_i+\phi_N\sum_j s_j\right)q_i-\frac{\overline{m}_q}{2}q_i^2,\quad \forall i,q_i,s, \tag{A.1}$$

with ϕ_N satisfying $A4^N$, for which the efficient allocation and price are defined as (see Example 1)

$$q_i^0(s)=\frac{\underline{m}_o-\phi_N}{\overline{m}_q}(s_i-\bar{s}),\quad p^0(s)=\left(\underline{m}_o+\phi_N(N-1)\right)\bar{s}. \tag{A.2}$$

Let the signal space S be such that A1 is satisfied. Note that for every s_{-i} the projections $\left\{s_i|\left(s_i,s_{-i}\right)\in S\right\}$ are closed intervals.

In this environment, the derivative of the Walrasian price with respect to WE quantity allocated to player 1 satisfies

$$\begin{aligned}\frac{dp^0\left(s_1^0(q_1),s_{-1}\right)}{dq_1}&=\frac{\frac{\partial p^0\left(s_1^0(q_1),s_{-1}\right)}{\partial s_1}}{\frac{\partial q_1^0\left(s_1^0(q_1),s_{-1}\right)}{\partial s_1}}=\frac{\left(\underline{m}_o+\phi_N(N-1)\right)}{\frac{1}{\overline{m}_q}\left(\underline{m}_o(N-1)-\phi_N(N-1)\right)}\\&=\frac{\overline{m}_q}{\underline{m}_o-\phi_N}\left(\frac{\underline{m}_o}{N-1}+\phi_N\right)=:p_q^{\max},\end{aligned} \tag{A.3}$$

where the function $s_1^0(x)$ is defined so that $q_1^0(s_1^0(x),s_{-1})=x$, for every x. (We show in the proof of Proposition 2 that $p_q^{\max}$ is also the upper bound on the derivative.)

29. Because we are not insisting on responsive allocation functions and strict ex post incentive compatibility, we may not use Theorem 2 in Bergemann and Morris (2009). However, the direct analysis in Section 3 of their paper applied to our linear setting implies that if the above condition is violated, an agent i and his beliefs over the types of his opponents exist such that all i's types are indistinguishable.

30. Alternative sufficient conditions for the global invertibility exist (see (38) in Dasgupta and Maskin (2000) and chapter 6 in Krantz and Parks (2012)).

Consider the signal profile s^*, with

$$s_i^* = \widetilde{s} > 0, \quad q_i^0(s^*) = \frac{N-1}{N}\overline{q}, \quad \forall i \leqslant N/2,$$

$$s_i^* = -\widetilde{s} < 0, \quad q_i^0(s^*) = -\frac{N-1}{N}\overline{q}, \quad \forall i > N/2.$$

The signal profile is chosen in such a way that for the function $s_1(x)$ defined as in (3.11), $\left(s_1(x), s_{-1}^*\right) \in S$, for every $x \in [0, q_1^0(s^*)]$.[31] Note also that $p^0(s^*) = 0$.

Fix $\varepsilon > 0$ and a mechanism that satisfies IR, IC, MC and is ε-efficient. The transfers by player 1 satisfy[32]

$$\begin{aligned}
t_1(s^*) &\leqslant \int_0^{q_1(s^*)} mu_1(x, s_1(x), s_{-1}^*)dx \\
&\leqslant \int_0^{q_1(s^*)} mu_1(x, \min\{s_1^*, s_1^0(x+\varepsilon)\}, s_{-1}^*)dx \\
&\leqslant \int_0^{q_1^0(s^*)-\varepsilon} mu_1(x, s_1^0(x+\varepsilon), s_{-1}^*)dx + \int_{q_1^0(s^*)-\varepsilon}^{q_1^0(s^*)} mu_1(x, s_1^*, s_{-1}^*)dx \\
&= \int_0^{q_1^0(s^*)-\varepsilon} \left[p^0(s_1^0(x+\varepsilon), s_{-1}^*) + \varepsilon\overline{m}_q\right]dx + \frac{1}{2}\overline{m}_q\varepsilon^2.
\end{aligned}$$

The first inequality follows from local IC and IR at $s_1(0)$, and the second one follows from ε-efficiency and monotonicity. Consequently,

$$\begin{aligned}
t_1(s^*) &\leqslant \int_\varepsilon^{q_1^0(s^*)} \left[p^0(s_1^0(x), s_{-1}^*) + \varepsilon\overline{m}_q\right]dx + \frac{1}{2}\overline{m}_q\varepsilon^2 \\
&= \int_\varepsilon^{q_1^0(s^*)} \left[p^0(s^*) + p_q^{\max} \times (x - q_1^0(s^*)) + \varepsilon\overline{m}_q\right]dx + \frac{1}{2}\overline{m}_q\varepsilon^2 \\
&= \varepsilon\overline{m}_q\left(q_1^0(s^*) - \varepsilon + \frac{1}{2}\varepsilon\right) - \frac{1}{2}(q_1^0(s^*) - \varepsilon)^2 p_q^{\max} \\
&\leqslant C_2 \times \varepsilon - C_1^N\left(\frac{m_o}{N-1} + \phi_N\right),
\end{aligned}$$

31. When, say, agent $1's$ signal is changed to $-\frac{1}{N-1}$, we have that

$$q_1^0(-\frac{1}{N-1}s_1^*, s_{-1}^*) = 0,$$

$$q_i^0(-\frac{1}{N-1}s_1^*, s_{-1}^*) = \overline{q}, \quad \forall i,j \leqslant N/2$$

$$|q_i^0(-\frac{1}{N-1}s_1^*, s_{-1}^*)| < \overline{q}. \quad \forall i,j > N/2$$

32. Note that the allocation $q_1(\cdot, s_{-1}^*)$ must be weakly increasing, from IC. If $q_1(\cdot, s_{-1}^*) = q_1'$ is constant over a range of signals $[\underline{s}, \overline{s}]$, then $s_1\left(q_1'\right)$ can be defined as any selection from $[\underline{s}, \overline{s}]$.

where, using the formula (A.3) for p_q^0,

$$C_1^N = \frac{1}{2}\left(\frac{N-1}{N}\overline{q}-\varepsilon\right)^2 \frac{\overline{m}_q}{\underline{m}_o-\phi_N}, \quad C_2=\overline{q}\overline{m}_q.$$

The same bound holds for the transfers by any buyer $i \leqslant \frac{N}{2}$. Analogous argument shows that the same bound on transfers holds for any seller $i > \frac{N}{2}$. This establishes the proof.

Appendix B: Proof of Proposition 2

Fix a slope $\sigma \geqslant 0$, profile of utility functions and the corresponding σ-VCG mechanism. By construction, the mechanism satisfies MC, so we only have to prove that it satisfies IR, IC, δ-BS and ε-Eff. We will do it in three steps.

1. The σ-VCG mechanism satisfies IR and IC.

Note that the σ-VCG transfers satisfy

$$t_i^\sigma(s)=\widetilde{t}_i\left(q_i^\sigma(s),s_{-i}\right),$$

for appropriate functions $\widetilde{t}_i$, $i \leqslant N$. Fix a type vector s and an agent i and consider the allocation $q_i^\sigma(\cdot,s_{-i})$ as a function of $i's$ signal. Local IC is equivalent to

$$mu_i\left(q_i^\sigma(s_i,s_{-i}),(s_i,s_{-i})\right)\frac{\partial q_i^\sigma(s_i,s_{-i})}{\partial s_i}=\frac{\partial \widetilde{t}_i\left(q_i^\sigma(s_i,s_{-i}),s_{-i}\right)}{\partial q_i}\frac{\partial q_i^\sigma(s_i,s_{-i})}{\partial s_i},$$

$$\frac{\partial \widetilde{t}_i\left(q_i^\sigma(s_i,s_{-i}),s_{-i}\right)}{\partial q_i}=mu_i\left(q_i^\sigma(s_i,s_{-i}),(s_i,s_{-i})\right),$$

and so for any $s_i' < s_i$

$$\begin{aligned}
t_i^\sigma(s)=\widetilde{t}_i\left(q_i^\sigma(s),s_{-i}\right)&=t_i^\sigma(s_i',s_{-i})+\int_{q_i^\sigma(s_i',s_{-i})}^{q_i^\sigma(s)}\frac{\partial \widetilde{t}_i(x,s_{-i})}{\partial q_i}dx\\
&=t_i^\sigma(s_i',s_{-i})+\int_{q_i^\sigma(s_i',s_{-i})}^{q_i^\sigma(s)} mu_i\left(x,\left(s_i(x),s_{-i}\right)\right)dx\\
&=t_i^\sigma(s_i',s_{-i})+\int_{q_i^\sigma(s_i',s_{-i})}^{q_i^\sigma(s)}\left[p^\sigma\left(s_i(x),s_{-i}\right)+\sigma x\right]dx,
\end{aligned}$$

which is precisely the definition of σ-VCG tranfers. This implies local IC.

Let us establish monotonicity of the allocation, $\frac{\partial q_i^\sigma(s)}{\partial s_i} \geqslant 0$, for all i and s. From the first order condition we have for any j and i

$$mu_{j,q}(q_i^\sigma(s),s)\frac{\partial q_j^\sigma(s)}{\partial s_i}+mu_{j,i}(q_j^\sigma(s),s)=\frac{\partial p^\sigma(s)}{\partial s_i}+\sigma\frac{\partial q_j^\sigma(s)}{\partial s_i}. \tag{B.1}$$

With $i=j$ this establishes that

$$\frac{\partial q_i^\sigma(s)}{\partial s_i}=\frac{mu_{i,i}(q_i^\sigma(s),s)-\frac{\partial p^\sigma(s)}{\partial s_i}}{\sigma-mu_{i,q}(q_i^\sigma(s),s)}. \tag{B.2}$$

On the other hand, since market clears for all s', it follows that

$$0=\sum_j \frac{\partial q_j^\sigma(s)}{\partial s_i}=\sum_j \frac{mu_{j,i}(q_j^\sigma(s),s)-\frac{\partial p^\sigma(s)}{\partial s_i}}{\sigma-mu_{j,q}(q_j^\sigma(s),s)},$$

and so

$$\frac{\partial p^\sigma(s)}{\partial s_i}=\frac{\sum_j mu_{j,i}(q_j^\sigma(s),s)\times(\sigma-mu_{j,q}(q_j^\sigma(s),s))^{-1}}{\sum_j(\sigma-mu_{j,q}(q_j^\sigma(s),s))^{-1}}. \tag{B.3}$$

Since the ratio of any two weights $(\sigma-mu_{j,q}(q_j^\sigma(s),s))^{-1}$ in formula (B.3) is at most $\frac{\overline{m}_q}{\underline{m}_q}$, formulas (B.2) and (B.3), together with $A2$ and $A4^N$ establish monotonicity. Given local IC and $mu_{i,i}(q_i,s)>0$ this implies full IC. By construction, IR is satisfied in each of the three (exclusive) cases: for the signal s_i under which $q_i^\sigma(s_i,s_{-i})=0$, signal $\inf S_i$ when $q_i^\sigma(S_i,s_{-i})>0$ and $\sup S_i$ when $q_i^\sigma(S_i,s_{-i})<0$. IC implies then IR at all signals, in each of the three cases.

2. σ-VCG satisfies δ-BS, for δ as in (3.12), and appropriate D_1^N,D_2^N.

Fix s,i, and q_i and let us write

$$y_j=\left(\sigma-\frac{\partial mu_j(q_j^\sigma(s_i(q_i),s_{-i}),s_i(q_i),s_{-i})}{\partial q_j}\right)^{-1},\quad \widetilde{y}_j=\frac{y_j}{\sum_{k=1}^N y_k}, \tag{B.4}$$

with $s_i(x)$ as in (3.11). Using formulas (B.2) and (B.3) we may bound the sensitivity of σ-Walrasian price with respect to $i's$ allocation:

$$\frac{dp^\sigma\left(s_i(q_i),s_{-i}\right)}{dq_i}=\frac{\frac{\partial p^\sigma(s_i(q_i),s_{-i})}{\partial s_i}}{\frac{\partial q_i^\sigma(s_i(q_i),s_{-i})}{\partial s_i}}=\frac{\sum_j mu_{j,i}(q_j^\sigma(s),s)\widetilde{y}_j}{\left(mu_{i,i}(q_i^\sigma(s),s)-\sum_j mu_{j,i}(q_j^\sigma(s),s)\widetilde{y}_j\right)y_i} \tag{B.5}$$

$$=\frac{mu_{i,i}(q_i^\sigma(s),s)}{mu_{i,i}(q_i^\sigma(s),s)\sum_{j\neq i}y_j-\sum_{j\neq i}mu_{j,i}(q_j^\sigma(s),s)y_j}$$

$$+\frac{\sum_{j\neq i}mu_{j,i}(q_j^\sigma(s),s)\widetilde{y}_j}{\left(mu_{i,i}(q_i^\sigma(s),s)\sum_{j\neq i}\widetilde{y}_j-\sum_{j\neq i}mu_{j,i}(q_j^\sigma(s),s)\widetilde{y}_j\right)y_i}$$

$$\leqslant\frac{\underline{m}_o}{(\underline{m}_o-\phi_N)\frac{N-1}{\sigma+\overline{m}_q}}+\frac{\phi_N}{(\underline{m}_o-\phi_N)\frac{1}{\sigma+\overline{m}_q}}=\frac{\sigma+\overline{m}_q}{\underline{m}_o-\phi_N}\left(\frac{\underline{m}_o}{N-1}+\phi_N\right).$$

Note that in the case of the Walrasian price p^0, the bound on the derivative is exactly $p_q^{\max}$, as in (A.3) in the proof of Proposition 1. Given this bound, the transfers satisfy

$$t_i^\sigma(s)=\int_0^{q_i^\sigma(s)}\left[p^\sigma\left(s_i(x),s_{-i}\right)+\sigma x\right]dx \tag{B.6}$$

$$\geqslant\int_0^{q_i^\sigma(s)}\left[p^\sigma(s)-\frac{\sigma+\overline{m}_q}{\underline{m}_o-\phi_N}\left(\frac{\underline{m}_o}{N-1}+\phi_N\right)(q_i^\sigma(s)-x)+\sigma x\right]dx$$

$$=p^\sigma(s)q_i(s)+\frac{q_i^\sigma(s)^2}{2}\left[\sigma-\frac{\sigma+\overline{m}_q}{\underline{m}_o-\phi_N}\left(\frac{\underline{m}_o}{N-1}+\phi_N\right)\right].$$

This implies that BS will be satisfied as long as $\sigma \geqslant \sigma_N$, for σ_N as in (3.13), and Assumption $A4^N$ is sufficient (and necessary, when $\frac{\underline{m}_q}{\overline{m}_q} \geqslant \frac{N-2}{2(N-1)}$) so that $\sigma_N > 0$ is well defined for every $N > 2$.

We now claim that for every $\sigma \geqslant 0, i$ and s we have $\left|q_i^\sigma(s)\right| \leqslant \overline{q}$. First, we have $p^\sigma(s) - \sigma\overline{q} \leqslant p^0(s) \leqslant p^\sigma(s) + \sigma\overline{q}$: otherwise, say, $p^\sigma(s) - \sigma\overline{q} > p^0(s)$ would imply $q_i^\sigma(s) < q_i^0(s)$, for every i, contradicting market clearing. This, in turn, implies that $\left|q_i^\sigma(s)\right| \leqslant \overline{q}$.

Consequently, (B.6) yields

$$t_i^\sigma(s) \geqslant p^\sigma(s) q_i(s) + \left[D_2^N \times \sigma - D_1^N\left(\frac{m_o}{N-1} + \phi_N\right)\right]_-, \tag{B.7}$$

where

$$D_1^N = \frac{1}{2}\overline{q}^2 \frac{\overline{m}_q}{\underline{m}_o - \phi_N}, \quad D_2^N = \frac{\overline{q}^2}{2}\left[1 - \frac{1}{\underline{m}_o - \phi_N}\left(\frac{m_o}{N-1} + \phi_N\right)\right].$$

Adding up the bounds (B.7) over all the agents establishes the proof of part 2.

3. σ-VCG mechanism satisfies ε-Eff, for ε as in (3.12).

Fix a σ-Walrasian equilibrium mechanism $\{(q^\sigma(s), t^\sigma(s))\}_{s\in S}$ and s. For any i we have

$$q_i^0(s) - q_i^\sigma(s) \leqslant \frac{1}{\underline{m}_q}\left[mu_i(q_i^\sigma(s), s) - mu_i(q_i^0(s), s)\right] \leqslant \frac{1}{\underline{m}_q}\left[\sigma q_i^\sigma(s) + p^\sigma(s) - p^0(s)\right]. \tag{B.8}$$

Below we provide two bounds for the difference $p^\sigma(s) - p^0(s)$. To establish the first bound, for any i define $q_i^\sigma(s,p)$ as the allocation that solves the first of the σ-VCG equations (3.8), but with p in place of $p^\sigma(s)$. Notice first that

$$q_i(s, p^0(s)) = q_i^0(s) \frac{mu_{q,i}^\#}{\sigma + mu_{q,i}^\#},$$

where $mu_{q,i}^\# \in [\underline{m}_q, \overline{m}_g]$ is such that

$$mu_{q,i}^\# = \frac{mu_i(q_i^0(s), s) - mu_i(q_i^\sigma(s, p^0(s)), s)}{q_i^\sigma(s, p^0(s)) - q_i^0(s)}.$$

Thus,

$$\left|\sum\nolimits_{i\leqslant N} q_i^\sigma(s, p^0(s))\right| = \left|\sum\nolimits_{i\leqslant N} q_i^0(s) \frac{mu_{q,i}^\#}{\sigma + mu_{q,i}^\#}\right| = \left|\sum\nolimits_{i\leqslant N} q_i^0(s) \frac{-\sigma}{\sigma + mu_{q,i}^\#}\right|$$
$$\leqslant \sigma\left(\frac{1}{\sigma + \underline{m}_q} - \frac{1}{\sigma + \overline{m}_q}\right)\frac{1}{2}\sum\nolimits_{i\leqslant N} |q_i^0(s)| \leqslant \frac{\sigma(\overline{m}_q - \underline{m}_q)}{(\sigma + \overline{m}_q)(\sigma + \underline{m}_q)} \frac{N\overline{q}}{2}.$$

Since on the other hand, it follows from the definition of $q_i^\sigma(s,p)$ that

$$\frac{\partial}{\partial p}\sum\nolimits_{i\leqslant N} q_i^\sigma(s,p) \geqslant -\frac{N}{\sigma + \overline{m}_q},$$

we have

$$\left|p^{\sigma}(s)-p^{0}(s)\right| \leqslant \frac{\frac{\sigma(\overline{m}_q-\underline{m}_q)}{(\sigma+\overline{m}_q)(\sigma+\underline{m}_q)}\frac{N\overline{q}}{2}}{\frac{N}{\sigma+\overline{m}_q}} = \frac{\sigma\overline{q}(\overline{m}_q-\underline{m}_q)}{2(\sigma+\underline{m}_q)}. \tag{B.9}$$

To establish the second bound, we claim that for the Walrasian price $p^0(s)$,

$$p^0(s) \in \left[\min_i mu_i(q_i^{\sigma}(s),s), \max_i mu_i(q_i^{\sigma}(s),s)\right].$$

This is because the allocation $q^{\sigma}(s)$ clears the market. If, say, $p^0(s)$ was above the highest $mu_i(q_i^{\sigma}(s),s)$ across all i, and so each agent i got less than $q_i^{\sigma}(s)$, market would not clear. The above inclusion establishes that

$$\left|p^{\sigma}(s)-p^{0}(s)\right| \leqslant \sigma\overline{q}. \tag{B.10}$$

Inequalities (B.8), (B.9), and (B.10) thus imply

$$\left|q_i^0(s)-q_i^{\sigma}(s)\right| \leqslant \frac{1}{\underline{m}_q}\left|\sigma\overline{q}+p^{\sigma}(s)-p^0(s)\right| \leqslant \frac{\sigma\overline{q}}{\underline{m}_q}\left[1+\min\left\{1,\frac{(\overline{m}_q-\underline{m}_q)}{2\underline{m}_q}\right\}\right],$$

which yields the second equality in (3.12).

Finally, note that *A*4 and *A*4^N imply that for σ_N defined in (3.13), $\sigma_N(\frac{1}{N}+\phi_N)^{-1}$ is uniformly bounded in N. This, together with part 3 of the proof implies the first part of the proposition.

Appendix C: Proofs for Section 4

Example 4. Proposition 2 (and the preceding discussion) in Rostek and Weretka (2012) implies that (using original notation)

$$\alpha = c_s = \frac{1-\rho}{1-\rho+\zeta},$$

$$\frac{\beta}{\alpha+\beta} = c_p = \frac{\left(2-\frac{N-2}{N-1}\right)\rho}{1-\frac{N-2}{N-1}+\rho}\,\frac{\zeta}{1-\rho+\zeta} = \frac{N\rho\zeta}{(1-\rho+\zeta)(1+(N-1)\rho)},$$

and so

$$\beta = \frac{\frac{N\rho\zeta}{(1-\rho+\zeta)(1+(N-1)\rho)}\frac{1-\rho}{1-\rho+\zeta}}{\frac{(1-\rho+\zeta)(1+(N-1)\rho)-N\rho\zeta}{(1-\rho+\zeta)(1+(N-1)\rho)}} = \frac{N\rho\zeta}{(1-\rho+\zeta)(1+\zeta+(N-1)\rho)}.$$

The submitted inverse demands are

$$\widetilde{p}(s_i)(\textstyle\sum_{j\neq i}q_j) = \frac{c_s}{1-c_p}s_i + \frac{\mu}{\frac{N-2}{N-1}-c_p}\textstyle\sum_{j\neq i}q_j = (\alpha+\beta)\left[s_i+\frac{\mu(N-1)}{\alpha(N-2)-\beta}\textstyle\sum_{j\neq i}q_j\right],$$

whereas the market clearing price $p^l(s)$ and allocation $q_i^l(s)$, for any s, satisfy

$$p^l(s) = \frac{c_s}{1-c_p}\overline{s} = (\alpha+\beta)\overline{s},$$

$$q_i^l(s) = \frac{c_s}{\mu}\left(\frac{\frac{N-2}{N-1}-c_p}{1-c_p}\right)(s_i-\overline{s}) = \frac{(N-2)\alpha-\beta}{\mu(N-1)}(s_i-\overline{s}),$$

which means that the transfers and the allocation is the same as in the σ_N-VCG mechanism (see (4.16)).

Example 5. Proposition 2 in Rostek and Weretka (2012) implies that for any s the equilibrium price $p^l(s)$ equals $\frac{c_s}{1-c_p}\bar{s}$, and the linear equilibrium has the form (4.19) for constants (using the original notation):

$$c_s = \frac{1-\frac{N-2}{2(N-1)}\rho}{1-\frac{N-2}{2(N-1)}\rho+\zeta},$$

$$c_p = \frac{\frac{N}{N-1}\frac{N-2}{2(N-1)}\rho}{\frac{1}{N-1}+\frac{N-2}{2(N-1)}\rho}\frac{\zeta}{1-\frac{N-2}{2(N-1)}\rho+\zeta} = \frac{\frac{N(N-2)}{2(N-1)}\rho}{1+\frac{N-2}{2}\rho}\frac{\zeta}{1-\frac{N-2}{2(N-1)}\rho+\zeta}.$$

Example 6. 1. With slope σ_N from Proposition 2 the transfers for $i \in G, G=S,B$, become:

$$t_i(s) = \int_0^{q_i(s)} \left[p^{\sigma_N}\left(s_i(x), s_{-i}\right)+\sigma_N x\right]dx = p^{\sigma}(s)q_i(s)$$
$$+\frac{q_i(s)^2}{2}\left[\sigma_N - \frac{\sigma_N+\mu_G}{\alpha(N-2\gamma_G)}(\beta+2\alpha\gamma_G)\right]$$
$$= p^{\sigma}(s)q_i(s)+\frac{q_i(s)^2}{2}\left[\sigma_N - \frac{\sigma_N+\mu_G}{\alpha(N-2\frac{\sigma_N+\mu_{-G}}{2\sigma_N+\mu_B+\mu_S})}\left(\beta+2\alpha\frac{\sigma_N+\mu_{-G}}{2\sigma_N+\mu_B+\mu_S}\right)\right].$$

2. Given a Bayesian model, the parameters $\sigma^2_\theta, \sigma^2_{\theta_G}, \sigma^2_{\theta_{id}}, \sigma^2_{\varepsilon_G}, \sigma^2_{\varepsilon_{id}}$ that give rise to utilities as in Example 3 are, from the Projection Theorem for Normal distributions,

$$\begin{bmatrix}\alpha\\ \beta/2\\ \beta/2\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} - Var(\varepsilon_S+\varepsilon_i, \bar{s}^S, \bar{s}^B)^{-1}_{3\times 3}\cdot Cov(\varepsilon_S+\varepsilon_i, (\varepsilon_S+\varepsilon_i, \bar{s}^S, \bar{s}^B)^T)_{3\times 1},$$

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = Var(\varepsilon_S+\varepsilon_i, \bar{s}^S, \bar{s}^B)_{3\times 3}\cdot\begin{bmatrix}1-\alpha\\ -\beta/2\\ -\beta/2\end{bmatrix} - Cov(\varepsilon_S+\varepsilon_i, (\varepsilon_S+\varepsilon_i, \bar{s}^S, \bar{s}^B)^T)_{3\times 1}$$
$$= \begin{bmatrix}A & B & \sigma^2_\theta\\ B & B & \sigma^2_\theta\\ \sigma^2_\theta & \sigma^2_\theta & B\end{bmatrix}\cdot\begin{bmatrix}1-\alpha\\ -\beta/2\\ -\beta/2\end{bmatrix} - \begin{bmatrix}\sigma^2_{\varepsilon_G}+\sigma^2_{\varepsilon_{id}}\\ \sigma^2_{\varepsilon_G}+\frac{2}{N}\sigma^2_{\varepsilon_{id}}\\ 0\end{bmatrix},$$

for $i \in S$ and similarly for $i \in B$, where

$$A = \sigma^2_\theta+\sigma^2_{\theta_G}+\sigma^2_{\theta_{id}}+\sigma^2_{\varepsilon_G}+\sigma^2_{\varepsilon_{id}},$$

$$B = \sigma^2_\theta+\sigma^2_{\theta_G}+\sigma^2_{\varepsilon_G}+\frac{\sigma^2_{\theta_{id}}+\sigma^2_{\varepsilon_{id}}}{N/2}.$$

It is readily verified that the solution to the above system of linear equations is given by (4.21).

3. For a fixed set of parameters $\sigma^2_\theta, \sigma^2_{\theta_G}, \sigma^2_{\theta_{id}}, \sigma^2_{\varepsilon_G}, \sigma^2_{\varepsilon_{id}}$ consider a linear Bayesian equilibrium in double auction of the form

$$q_i^l(s_i)(p) = a_G s_i - b_G p, \quad \forall i \in G, G = S, B,$$

and so the equilibrium price p^l is given by

$$p^l(s) = \frac{a_S}{b_S + b_B}\bar{s}^S + \frac{a_B}{b_S + b_B}\bar{s}^B. \tag{C.1}$$

From the first-order condition, demand schedules are given by equation (4.22), for

$$d_i = \frac{1}{\frac{N-2}{2}b_G + \frac{N}{2}b_{-G}}, \quad \forall i \in G, G = S, B. \tag{C.2}$$

From the Projection Theorem, we compute $E[\theta_i|s_i, p]$, for $i \in G$, as

$$E\left[\theta_i|s_i, p^l\right] = \widetilde{a}_G s_i - \widetilde{b}_G p^l, \tag{C.3}$$

where

$$\begin{bmatrix} \widetilde{a}_G \\ -\widetilde{b}_G \end{bmatrix} = \text{Var}(s_i, p^l)^{-1}_{2\times 2} \cdot Cov(\theta_i, (s_i, p^l)^T)_{2\times 1},$$

$$\text{Var}((s_i, p^l))_{2\times 2} =$$

$$= \begin{bmatrix} \sigma^2_\theta + \sigma^2_{\theta_G} + \sigma^2_{\theta_{id}} + \sigma^2_{\varepsilon_G} + \sigma^2_{\varepsilon_{id}} & \sigma^2_\theta + \frac{a_G}{b_G + b_{-G}}\left(\sigma^2_{\theta_G} + \sigma^2_{\varepsilon_G} + \frac{2\left(\sigma^2_{\theta_{id}} + \sigma^2_{\varepsilon_{id}}\right)}{N}\right) \\ \sigma^2_\theta + \frac{a_G}{b_G + b_{-G}}\left(\sigma^2_{\theta_G} + \sigma^2_{\varepsilon_G} + \frac{2\left(\sigma^2_{\theta_{id}} + \sigma^2_{\varepsilon_{id}}\right)}{N}\right) & \sigma^2_\theta + \frac{a_G^2 + a_{-G}^2}{(b_G + b_{-G})^2}\left(\sigma^2_{\theta_G} + \sigma^2_{\varepsilon_G} + \frac{2\left(\sigma^2_{\theta_{id}} + \sigma^2_{\varepsilon_{id}}\right)}{N}\right) \end{bmatrix},$$

$$\text{Cov}(\theta_i, (s_i, p^l)^T)_{2\times 1} = \begin{bmatrix} \sigma^2_\theta + \sigma^2_{\theta_G} + \sigma^2_{\theta_{id}} \\ \sigma^2_\theta + \frac{a_G}{b_G + b_{-G}}\left(\sigma^2_{\theta_G} + \frac{2}{N}\sigma^2_{\theta_{id}}\right) \end{bmatrix}.$$

Substituting (4.21) into these formulas, we get

$$\begin{aligned} \tilde{a}_G = & \frac{2(\alpha-1)\alpha\left(a_G^2(N-2)\sigma_{\varepsilon_{id}} + a_{-G}^2 N(\sigma_{\varepsilon_{id}} + \sigma_{\varepsilon_G})\right)}{2(\alpha-1)\left(a_G^2(N-2)\sigma_{\varepsilon_{id}} + a_{-G}^2 N(\sigma_{\varepsilon_{id}} + \sigma_{\varepsilon_G})\right) + \beta(N-2)\sigma_{\varepsilon_{id}}(a_G - a_{-G})^2} \\ & + \frac{\beta(a_G - a_{-G})(\alpha a_G(N-2)\sigma_{\varepsilon_{id}} + a_{-G}(-\alpha N(\sigma_{\varepsilon_{id}} + \sigma_{\varepsilon_G}) + n\sigma_{\varepsilon_G} + 2\sigma_{\varepsilon_{id}}))}{2(\alpha-1)\left(a_G^2(N-2)\sigma_{\varepsilon_{id}} + a_{-G}^2 N(\sigma_{\varepsilon_{id}} + \sigma_{\varepsilon_G})\right) + \beta(N-2)\sigma_{\varepsilon_{id}}(a_G - a_{-G})^2}, \\ \tilde{b}_G = & \frac{(\alpha-1)\beta(b_G + b_{-G})(a_G(N-2)\sigma_{\varepsilon_{id}} + a_{-G}N(\sigma_{\varepsilon_{id}} + \sigma_{\varepsilon_G}))}{2(\alpha-1)\left(a_G^2(N-2)\sigma_{\varepsilon_{id}} + a_{-G}^2 N(\sigma_{\varepsilon_{id}} + \sigma_{\varepsilon_G})\right) + \beta(N-2)\sigma_{\varepsilon_{id}}(a_G - a_{-G})^2}. \end{aligned}$$

Equilibrium is thus characterized by the solutions a_S, a_B, b_S, b_B of the system below:

$$a_G = \frac{\widetilde{a}_G(a_G, a_{-G})}{\mu_G + d_G(b_G, b_{-G})}, \quad b_G = \frac{\widetilde{b}_G(a_G, a_{-G}, b_G + b_{-G}) + 1}{\mu_G + d_G(b_G, b_{-G})}, \quad G = S, B. \tag{C.4}$$

Note that $a_G \neq a_{-G}$ as long as $\mu_G \neq \mu_{-G}$:

$$\frac{a_G}{a_{-G}} = 1 \Rightarrow \frac{b_G}{b_{-G}} = 1 \Rightarrow \mu_G = \mu_{-G}.$$

In the asymmetric case, (C.4) is a system of four equations of third order and so generically does not admit an analytical solution (Abel–Ruffini Theorem).

4. Finally, to see that the solution to the system (C.4) depends locally on the parameters $\sigma_{\varepsilon_{id}}$ and σ_{ε_G}, even when α, β are fixed, observe that:

$$\frac{\partial \tilde{a}_G}{\partial \sigma^2_{\varepsilon_{id}}} = \frac{(\alpha-1)\beta a_{-G}(N-2)N\sigma_{\varepsilon_G}(a_G-a_{-G})\big(2(\alpha-1)\big(a_G^2+a_{-G}^2\big)+\beta(a_G-a_{-G})^2\big)}{\big(2(\alpha-1)\big(a_G^2(N-2)\sigma_{\varepsilon_{id}}+a_{-G}^2N(\sigma_{\varepsilon_{id}}+\sigma_{\varepsilon_G})\big)+\beta(N-2)\sigma_{\varepsilon_{id}}(a_G-a_{-G})^2\big)^2} \neq 0,$$

$$\frac{\partial \tilde{a}_G}{\partial \sigma^2_{\varepsilon_G}} = -\frac{(\alpha-1)\beta a_{-G}(N-2)N\sigma_{\varepsilon_{id}}(a_G-a_{-G})\big(2(\alpha-1)\big(a_G^2+a_{-G}^2\big)+\beta(a_G-a_{-G})^2\big)}{\big(2(\alpha-1)\big(a_G^2(N-2)\sigma_{\varepsilon_{id}}+a_{-G}^2N(\sigma_{\varepsilon_{id}}+\sigma_{\varepsilon_G})\big)+\beta(N-2)\sigma_{\varepsilon_{id}}(a_G-a_{-G})^2\big)^2} \neq 0,$$

as long as $N>2, \alpha \neq 1, \beta \neq 0, \sigma_{\varepsilon_{id}} > 0, \sigma_{\varepsilon_G} > 0$, and $a_G \neq a_{-G} \neq 0$, which is true when $\mu_G \neq \mu_{-G}$.

Appendix D: Proofs for Section 5

Proof of Proposition 3. Fix $(u_1,...,u_N)$ and $\sigma \geqslant 0$. For the successful inference of the vector of signals from the price-allocation vector we have to show that $\varphi : \mathbb{R}^N \to \mathbb{R} \times \mathbb{B}$,

$$\varphi(s) = (p^{\sigma}(s), q^{\sigma}(s)),$$

is globally invertible (*i.e.* a diffeomorphism), where $\mathbb{B} \subset \mathbb{R}^N$ is a hyperplane given MC.

First, we will show that for any fixed $q \in \mathbb{R}^N$ the mapping $\zeta_q : \mathbb{R}^N \to \mathbb{R}^N$:

$$\zeta_q(s) = (mu_1(q_1,s),...,mu_N(q_N,s)),$$

is a diffeomorphism. By the global invertibility theorems (see Theorems 6.2.3 and 6.2.4 in Krantz and Parks (2012)) it is sufficient that (i) the mapping acts on the whole Euclidean space, which is our Assumption INF (ii) the mapping is non-degenerate (Jacobian determinant non-zero) and proper (Jacobian determinant uniformly bounded away from infinity), which is our Assumption INJ (iii) the mapping has a fixed point $\varphi(0)=0$, which is not true for $\zeta_q(s)$ but is true for $\zeta_q(s)-\zeta_q(0)$ and therefore has no bite.

Second, the mapping $\xi : \mathbb{R}^N \to \mathbb{R} \times \mathbb{B}$, where $\xi(z)=(p,q)$ solves $z_i = p + \sigma q_i$ for all i and $\sum_i q_i = 0$ is a diffeomorphism as well. The rank condition is easily verified:

$$\xi(z) = \begin{bmatrix} \frac{1}{N} \cdot e^T \\ \frac{1}{\sigma} \cdot I - \frac{1}{N} \cdot ee^T \end{bmatrix} \cdot z^T, \quad rank(\xi) = rank \begin{bmatrix} \frac{1}{N} \cdot e^T \\ \frac{1}{\sigma} \cdot I \end{bmatrix} = N,$$

where I is the $N \times N$ identity matrix and e is a $N \times 1$ unit vector.

Finally, assuming that for two signal profiles s, s': $\varphi(s) = \varphi(s') = (p,q)$:

$$s = \zeta_q^{-1}(\xi^{-1}(p,q)) = s',$$

which completes the proof that φ is globally invertible (*i.e.* a diffeomorphism).

With global invertibility at hand, we may verify that that at every s the profile of σ-VCG conditional inverse demands yields $(p^{\sigma}(s), q^{\sigma}(s))$ as the unique market clearing price-allocation pair. We will use an alternative representation of a conditional inverse demand correspondence d_i, via the set $D(d_i)$ of market clearing price-allocation pairs consistent with the trader's private signal (see Perry and Reny, 2002):

$$D_i(d_i) = \bigcup_{q_{-i}} (d_i(q_{-i}), -\sum_{j \neq i} q_j, q_{-i}).$$

Market clearing price-allocation pairs are then represented by the intersection $\bigcap_i D_i(d_i)$, which is informationally equivalent to the whole vector of private signals due to the fact that φ commutes with set operation $\cap$ under global invertibility:

$$\bigcap_i D_i(d_i) = \bigcap_i \varphi(A_i) = \varphi(\bigcap_i A_i) = \varphi(s), \quad A_i = \{s \in \mathbb{R}^n : s_i \text{ is fixed at true signal}\}.$$

Fix s and a player i. If all the players $j \neq i$ play the equilibrium strategies, the graph of any residual demand curve for a player is given by

$$\bigcap_{j \neq i} D_j(d_j) = \bigcap_{j \neq i} \varphi(A_j) = \varphi(\bigcap_{j \neq i} A_j) = \varphi(B_i), \quad B_i = \{s \in \mathbb{R}^n : s_{-i} \text{ is fixed at true signal}\}.$$

In other words the unique residual demand curve that i is facing is given by

$$p_i(q_i) = p^{\sigma}(s_i', s_{-i}), \text{ for } s_i' \text{ such that } q_i = q_i^{\sigma}(s_i', s_{-i}).$$

Given the definition of transfers, at s agent i is choosing between either i) a zero quantity and transfer pair (when he submits an incompatible conditional inverse demand function) or 2) pairs $(q_i^{\sigma}(s_i', s_{-i}), t_i^{\sigma}(s_i', s_{-i}))$, for some s_i', when he submits the σ-VCG conditional inverse demand at s_i'. The result thus follows from IC and IR of the σ-VCG mechanism.

Proof of Lemma 2 Fix i and s_i. As in the proof of Proposition 3, $\varphi_{i,s_i} : S_{-i} \to \mathbb{R}^N$ defined as $\varphi_{i,s_i}(s_{-i}) = (p^{\sigma}(s_i, s_{-i}), q_{-i}^{\sigma}(s_i, s_{-i}),)$ is injective. We have

$$\begin{aligned} d_i(s_i)(q_{-i}) &= \left\{x | x = p^{\sigma}(s_i, s_{-i}) \text{ and } q_{-i}^{\sigma}(s_i, s_{-i}) = q_{-i}\right\} \\ &= \left\{x | x = mu_i(-\sum_{j \neq i} q_j, s_i, s_{-i}) + \sigma \sum_{j \neq i} q_j \text{ and } q_{-i}^{\sigma}(s_i, s_{-i}) = q_{-i}\right\} \\ &= \left\{x | x = m\widetilde{u}_i(-\sum_{j \neq i} q_j, s_i, w_i(s_{-i})) + \sigma \sum_{j \neq i} q_j \text{ and } q_{-i}^{\sigma}(s_i, s_{-i}) = q_{-i}\right\} \\ &= \left\{x | x = m\widetilde{u}_i(-\sum_{j \neq i} q_j^{\sigma}, s_i, w_i(\varphi_{i,s_i}^{-1}(x, q_{-i}^{\sigma}))) + \sigma \sum_{j \neq i} q_j^{\sigma} \text{ and } q_{-i}^{\sigma} = q_{-i}\right\}, \end{aligned}$$

and so the result follows for $\widetilde{w}_i = w_i \circ \varphi_{i,s_i}^{-1}$.

If σ-VCG inverse demands are functions then the proof follows the same steps as above, with $\widetilde{w}_i = w_i \circ g_{i,s_i}^{-1}$ and $g_{i,s_i} : S_{-i} \to \mathbb{R}^{N-1}$, $g_i(s_{-i}) = q_{-i}^{\sigma}(s_i, s_{-i})$.

Proof of Lemma 3. Fix $\sigma \geqslant 0$ and let

$$J(s) = \left[mu_{i,j}(q_i^{\sigma}(s), s)\right]_{i,j \leqslant N}.$$

be the $N \times N$ Jacobian of the function $\zeta_q(s) = (mu_1(q_1, s), \ldots, mu_N(q_N, s))$ for $q = q^{\sigma}(s)$.

To prove the first part of the lemma, we totally differentiate equations (3.8) and invert $J(s)$ by Assumption INJ to get the following system of equations:

$$ds = J(s)^{-1}\left(\sigma I - \left[mu_{i,q}(q_i^\sigma, s)\right]_{i \leqslant N} \cdot I\right) dq + J(s)^{-1} e \cdot dp, \tag{D.1}$$

where I is the $N \times N$ identity matrix and e is the unit vector.

Consider two points $s = \varphi^{-1}(q^*, p)$ and $s' = \varphi^{-1}(q^*, p')$ such that $p < p'$, and a path

$$\gamma(t) = \varphi^{-1}(q^*, (1-t)p + tp'), \quad \gamma(0) = s, \quad \gamma(1) = s'$$

connecting these two points in the space of signals. Along this path $dq = 0$, therefore:

$$ds = J(s)^{-1} e \cdot dp,$$

and so, since dp does not change sign along the path and from Assumption FUN, it is also true that ds_k does not change sign along the path. Therefore, the signals are different in each coordinate at $\gamma(0)$ and $\gamma(1)$.

We have demonstrated the following property of the φ mapping. For two points $s = \varphi^{-1}(q^*, p)$ and $s' = \varphi^{-1}(q^*, p')$ such that $p < p'$, it is true that $s_i \neq s_i'$ for all i. Consequently, the demand correspondence is a demand function.

Suppose now that utilities are linear and $\left(J(0)^{-1} e\right)_k = 0$ for some agent k. It follows that for a vector $s = J(0)^{-1} e$, we have

$$mu(0,s) = mu(0,0) + J(0) \cdot s = mu(0,0) + e,$$

and so for any $\sigma \geqslant 0$, from the definition of σ-VCG mechanism,

$$q^\sigma(s) = q^\sigma(0),$$
$$p^\sigma(s) = p^\sigma(0) + 1,$$

implying that the σ-VCG conditional inverse demand of agent k at $s_k = 0$ is a correspondence.

Proof of Lemma 4

(i) Let matrices A and a be the diagonal and off-diagonal parts of matrix $\left[mu_{i,j}(0,0)\right]_{i,j \leqslant N}$, and let $b = \left[mu_{i,j}(0,0)\right]_{i,j \leqslant N}^{-1} - A^{-1}$. We have

$$(A^{-1} + b)(A + a) = I,$$
$$A^{-1} a + bA + ba = 0,$$
$$aA^{-1} + Ab(1 + aA^{-1}) = 0,$$
$$Ab = -aA^{-1}(1 + aA^{-1})^{-1}.$$

Since the infinity norm is both sub-multiplicative and sub-additive:

$$||Ab||_\infty \leqslant \frac{||aA^{-1}||_\infty}{1 - ||aA^{-1}||_\infty}.$$

Finally, by assumption, each element of the aA^{-1} matrix is less then $1/2n$, therefore $||aA^{-1}||_\infty < 1/2$. By the inequality above $||Ab||_\infty < 1$ and therefore matrix $W^{-1} = A^{-1}(I + Ab)$ has positive row sums, since A^{-1} has only positive diagonal elements.

(ii) Equicommonality means that $\left[mu_{i,j}(0,0)\right]_{i,j\leqslant N}\cdot e=[\alpha]_{N\times 1}$, for unit vector e and some $\alpha\neq 0$. Left multiplying by $\left[mu_{i,j}(0,0)\right]_{i,j\leqslant N}^{-1}$ yields the result.

Appendix E: Proof of Proposition 4

Consider a family of τ_N-VCG mechanisms that are robustly asymptotically $o\left(\sqrt{\frac{1}{N}+\phi_N}\right)$-efficient. Suppose that the condition (6.29) is violated, and so there is $q\in(0,\overline{q})$ such that

$$\left(\frac{\underline{m}_o}{N-1}+\phi_N\right)^{-1}\int_0^q\int_0^x \tau_N''(s)\,ds\,dx-\frac{\overline{m}_q}{(\underline{m}_o-\phi_N)}\frac{q^2}{2}\leqslant -\varepsilon<0. \tag{E.1}$$

We will establish that for sufficiently large N the mechanisms may violate BS.

Consider utility functions as in (A.1), large N (to be determined later) and a profile of signals $s^*=(s_1^*,s_2^*,0,\ldots,0)$ such that $s_2^*=-s_1^*$ and

$$mu_1(q,s^*)=s_1^*\underline{m}_o+\phi_N s_2^*-\underline{m}_q q=\tau_N'(q),$$

so that $q^{\tau_N}(s^*)=(q,-q,0,\ldots,0)$. Let $a_N=\lim_{q\to 0}\tau_N(q)-\tau_N(0)\geqslant 0$ be the entry fee of the tax and, with slight abuse of notation, let $\tau_N'(0)=\lim_{q\to 0}\tau_N'(q)$ be the initial slope of the tax scheme. As in Proposition 2 the transfers satisfy

$$t_1(s^*)\leqslant\int_0^q p^{\tau_N}(s_1(x),s_{-1}^*)dx+\tau_N(q),$$

$$t_2(s^*)\leqslant\int_0^{-q} p^{\tau_N}(s_2(x),s_{-2}^*)dx+\tau_N(q),$$

$$t_i(s^*)\leqslant 0,\ \ \forall i>2,$$

where $p^{\tau_N}(s)$ are defined, analogously to σ-VCG prices, as the Walrasian prices for the economy with distorted utilities $\widetilde{u}_i$, and $s_1(x)$ and $s_2(x)$ are defined analogously as in (3.11):

$$mu_i(q_i^{\tau_N}(s),s)=p^{\tau_N}(s)+\tau_N'(q_i^{\tau_N}(s)),\ \ \forall i,s,$$

$$x=q_i^{\tau_N}\left(s_i(x),s_{-i}^*\right),\ \ i=1,2,\forall x\in[-q,q].$$

Below we establish that $t_1(s^*)<0$; as the proof of $t_2(s^*)>0$ is analogous, this will establish the result.

Let s_1' and s_1'' be the signals for player 1, $s_1''\leqslant s_1'\leqslant s_1^*$, such that

$$\max_{q.}\{u_3(q.,s_1',s_{-i}^*)-q.\times p^{\tau_N}(s_1',s_{-i}^*)-a_N\}=0, \tag{E.2}$$

$$\max_{q.}\{u_3(q.,s_1'',s_{-i}^*)-q.\times p^{\tau_N}(s_1'',s_{-i}^*)-\tau_N(q.)\}=0.$$

In other words, if the tax scheme consisted of only the entry fee a_N then agents $i>2$ would start trading (buying from agent 2) if 1's signal drops below s_1'. Similarly, agents $i>2$ start trading given the original tax scheme τ_N when 1's signal drops below s_1''.

Let us compute the (negative) prices $p^{\tau_N}(s_1(x),s^*_{-1})$ for the inframarginal units x, $x\in[0,q]$. For $s_1\geqslant s''_1$ only players 1 and 2 trade non-zero quantities, and we have

$$mu_1(q_1^{\tau_N}(s_1,s^*_{-1}),s_1,s^*_{-1})=s_1\underline{m}_o+\phi_N s^*_2-\overline{m}_q q_1^{\tau_N}(s_1,s^*_{-1})=p^{\tau_N}(s_1,s^*_{-1})+\tau'_N(q_1^{\tau_N}(s_1,s^*_{-1})),$$
$$mu_1(q_2^{\tau_N}(s_1,s^*_{-1}),s_1,s^*_{-1})=s^*_2\underline{m}_o+\phi_N s_1+\overline{m}_q q_2^{\tau_N}(s_1,s^*_{-1})=p^{\tau_N}(s_1,s^*_{-1})+\tau'_N(q_2^{\tau_N}(s_1,s^*_{-1})),$$

and so

$$\frac{\partial q_1^{\tau_N}(s_1,s^*_{-1})}{\partial s_1}=\frac{\underline{m}_o-\frac{\partial p^{\tau_N}(s_1,s^*_{-1})}{\partial s_1}}{\overline{m}_q+\tau''_N(q_1^{\tau_N}(s_1,s^*_{-1}))},$$
$$\frac{\partial q_2^{\tau_N}(s_1,s^*_{-1})}{\partial s_1}=\frac{\phi_N-\frac{\partial p^{\tau_N}(s_1,s^*_{-1})}{\partial s_1}}{\overline{m}_q+\tau''_N(q_2^{\tau_N}(s_1,s^*_{-1}))}.$$

Using MC, we thus have

$$\frac{\partial p^{\tau_N}(s_1,s^*_{-1})}{\partial s_1}=\frac{\frac{\underline{m}_o}{\overline{m}_q+\tau''_N(q_1^{\tau_N}(s_1,s^*_{-1}))}+\frac{\phi_N}{\overline{m}_q+\tau''_N(q_2^{\tau_N}(s_1,s^*_{-1}))}}{\frac{1}{\overline{m}_q+\tau''_N(q_1^{\tau_N}(s_1,s^*_{-1}))}+\frac{1}{\overline{m}_q+\tau''_N(q_2^{\tau_N}(s_1,s^*_{-1}))}}=\frac{\underline{m}_o+\phi_N}{2},$$

and so, for any x such that $s_1(x)\geqslant s''_1$

$$\frac{\partial p^{\tau_N}(s_1(x),s^*_{-1})}{\partial x}=-\frac{\frac{\partial p^{\tau_N}(s_1,s^*_{-1})}{\partial s_1}}{\frac{\partial q_1^{\tau_N}(s_1,s^*_{-1})}{\partial s_1}}=\frac{\frac{\underline{m}_o+\phi_N}{2}}{\frac{\underline{m}_o-\phi_N}{2(\overline{m}_q+\tau''_N(x))}}=\left(\overline{m}_q+\tau''_N(x)\right)\frac{\underline{m}_o+\phi_N}{\underline{m}_o-\phi_N}. \tag{E.3}$$

Similarly, for $s_1<s''_1$ and $x\geqslant 0$ such that $s_1(x)<s''_1$, when all agents trade nonzero quantities, we have

$$\frac{\partial p^{\tau_N}(s_1,s^*_{-1})}{\partial s_1}=w_1(s_1)\underline{m}_o+(1-w_1(s_1))\phi_N, \tag{E.4}$$
$$\frac{\partial q_1^{\tau_N}(s_1,s^*_{-1})}{\partial s_1}=\frac{\underline{m}_o-\frac{\partial p^{\tau_N}(s_1,s^*_{-1})}{\partial s_1}}{\overline{m}_q+\tau''_N(q_1^{\tau_N}(s_1,s^*_{-1}))},$$
$$\frac{\partial p^{\tau_N}(s_1(x),s^*_{-1})}{\partial x}=\frac{\overline{m}_q+\tau''_N(x)}{(1-w_1(s_1(x)))(\underline{m}_o-\phi_N)}\left(w_1(s_1(x))\underline{m}_o+(1-w_1(s_1(x)))\phi_N\right),$$

where

$$w_1(s_1)=\frac{\frac{1}{\overline{m}_q+\tau''_N(q_1^{\tau_N}(s_1,s^*_{-1}))}}{\frac{1}{\overline{m}_q+\tau''_N(q_1^{\tau_N}(s_1,s^*_{-1}))}+\frac{1}{\overline{m}_q+\tau''_N(q_2^{\tau_N}(s_1,s^*_{-1}))}+\frac{N-2}{\overline{m}_q+\tau''_N(q_3^{\tau_N}(s_1,s^*_{-1}))}}.$$

Note that $o\left(\sqrt{\frac{1}{N}+\phi_N}\right)$-efficiency implies that the derivative τ'_N is bounded by a term of order $o\left(\sqrt{\frac{1}{N}+\phi_N}\right)$ on $\left[0,\overline{q}-o\left(\sqrt{\frac{1}{N}+\phi_N}\right)\right]$. Since also $\tau'_N(0)\geqslant 0$ and τ'_N is increasing, it follows that $\tau_N''(x)$ is bounded above by a term of order $o(1)$ on a subset of $[0,q]$ of measure $q-o\left(\sqrt{\frac{1}{N}+\phi_N}\right)$. It thus follows that $w_1(s_1)$ is bounded above by $(1+o(1))\frac{1}{N}$, and so, from (E.4),

$$\frac{\partial p^{\tau_N}(s_1(x),s_{-1}^*)}{\partial x} \geqslant \left(\frac{\overline{m}_q}{(\underline{m}_o-\phi_N)}-o(1)\right)\left(\frac{\underline{m}_o}{N-1}+\phi_N\right), \tag{E.5}$$

on a subset of $[0,q]$ of measure $q-o\left(\sqrt{\frac{1}{N}+\phi_N}\right)$, for every N.

Moreover, $o\left(\sqrt{\frac{1}{N}+\phi_N}\right)$-efficiency of the allocation for agents $i>2$ implies that

$$\tau_N'(0), p^{\tau_N}(s_1'',s_{-i}^*), p^{\tau_N}(s_1',s_{-i}^*) = o\left(\sqrt{\frac{1}{N}+\phi_N}\right), \tag{E.6}$$

and so, from (E.3),

$$q_1^{\tau_N}(s_1',s_{-1}^*), q_1^{\tau_N}(s_1'',s_{-1}^*) = q-o\left(\sqrt{\frac{1}{N}+\phi_N}\right).$$

Finally, since $\tau_N'\geqslant 0$ and $u_3(q_3,s_1'',s_{-i}^*)\leqslant u_3(q_3,s_1',s_{-i}^*)$ for every q_3, equations in (E.2) imply that

$$p^{\tau_N}(s_1',s_{-i}^*)-p^{\tau_N}(s_1'',s_{-i}^*)\geqslant \tau_N'(0).$$

The integral of the inframarginal prices agent 1 pays is

$$\int_0^q p^{\tau_N}(s_1(x),s_{-1}^*)dx = I+II+III,$$

$$I = q_1^{\tau_N}(s_1',s_{-1}^*)\times p^{\tau_N}(s_1',s_{-1}^*)+\int_{q_1^{\tau_N}(s_1',s_{-1}^*)}^{q} p^{\tau_N}(s_1(x),s_{-1}^*)dx,$$

$$II = q_1^{\tau_N}(s_1'',s_{-1}^*)\times\left(p^{\tau_N}(s_1'',s_{-1}^*)-p^{\tau_N}(s_1',s_{-1}^*)\right)$$
$$+\int_{q_1^{\tau_N}(s_1'',s_{-1}^*)}^{q_1^{\tau_N}(s_1',s_{-1}^*)} p^{\tau_N}(s_1(x),s_{-1}^*)-p^{\tau_N}(s_1',s_{-1}^*))dx,$$

$$III = \int_0^{q_1^{\tau_N}(s_1'',s_{-1}^*)} \left(p^{\tau_N}(s_1(x),s_{-1}^*)-p^{\tau_N}(s_1'',s_{-1}^*)\right)dx.$$

Below, we bound from above each of the three terms I, II, and III.

First, since

$$a_N = u_3(q_\cdot',s_1',s_{-i}^*)-q_\cdot'\times p^{\tau_N}(s_1',s_{-i}^*) = \frac{p^{\tau_N}(s_1',s_{-i}^*)^2}{2\overline{m}_q},$$

where $q_\cdot'$ is the maximizer in the first line of (E.2), we have, for N sufficiently high (see (E.6))

$$I < q_1^{\tau_N}(s_1',s_{-1}^*)\times p^{\tau_N}(s_1',s_{-1}^*) = \left(q-o\left(\sqrt{\frac{1}{N}+\phi_N}\right)\right)\times p^{\tau_N}(s_1',s_{-1}^*) < a_N.$$

Second,

$$II = q_1^{\tau_N}(s_1', s_{-1}^*) \times \left(p^{\tau_N}(s_1'', s_{-1}^*) - p^{\tau_N}(s_1', s_{-1}^*)\right) + \int_{q_1^{\tau_N}(s_1'', s_{-1}^*)}^{q_1^{\tau_N}(s_1', s_{-1}^*)} p^{\tau_N}(s_1(x), s_{-1}^*) - p^{\tau_N}(s_1'', s_{-1}^*)dx$$

$$= q \times \left(p^{\tau_N}(s_1'', s_{-1}^*) - p^{\tau_N}(s_1', s_{-1}^*)\right) + o\left(\frac{1}{N} + \phi_N\right) < -\tau_N' q + o\left(\frac{1}{N} + \phi_N\right).$$

Finally, the bound (E.5) yields

$$III = \int_0^{q_1^{\tau_N}(s_1'', s_{-1}^*)} \left(p^{\tau_N}(s_1(x), s_{-1}^*) - p^{\tau_N}(s_1'', s_{-1}^*)\right)dx$$

$$< \left(-\frac{\overline{m}_q}{(\underline{m}_o - \phi_N)} + o(1)\right)\left(\frac{\underline{m}_o}{N-1} + \phi_N\right) \times \frac{1}{2}\left(q_1^{\tau_N}(s_1'', s_{-1}^*) - o\left(\sqrt{\frac{1}{N} + \phi_N}\right)\right)^2$$

$$= \left(-\frac{\overline{m}_q}{(\underline{m}_o - \phi_N)} + o(1)\right)\frac{q^2}{2}\left(\frac{\underline{m}_o}{N-1} + \phi_N\right) + o\left(\frac{1}{N} + \phi_N\right).$$

The three bounds imply

$$t_1(s^*) = I + II + III + \tau_N(q) = I + a_N + II + \tau_N'(0) \times q + III + \int_0^q \int_0^x \tau_N''(s)dsdx$$

$$< \left(o(1) - \frac{\overline{m}_q}{(\underline{m}_o - \phi_N)}\right)\frac{q^2}{2}\left(\frac{\underline{m}_o}{N-1} + \phi_N\right) + \int_0^q \int_0^x \tau_N''(s)dsdx + o\left(\frac{1}{N} + \phi_N\right)$$

$$\leqslant \left(\frac{\underline{m}_o}{N-1} + \phi_N\right)\frac{q^2}{2}\left[o(1) - \frac{\overline{m}_q}{(\underline{m}_o - \phi_N)} + \frac{\overline{m}_q}{(\underline{m}_o - \phi_N)} - \varepsilon\right] + o\left(\frac{1}{N} + \phi_N\right),$$

which is strictly negative for N large enough. It follows that for N sufficiently high the transfers $t_1(s^*)$ are negative, which establishes the proof.

Appendix F: Proof of Proposition 5

In order to establish robust asymptotic $\frac{\sigma_N}{\overline{m}_q}$-$q$_efficient, IR, IC, and BS follow from Proposition 2. Moreover, for any utility functions, i and s the efficiency loss is bounded by

$$\left|q_i^\sigma(s) - q_i^0(s)\right| \leqslant \frac{1}{\overline{m}_q}\left|mu_i(q_i^\sigma(s), s) - mu_i(q_i^0(s), s)\right| = \frac{\sigma_N}{\overline{m}_q}\left|q_i^\sigma(s)\right|,$$

where the last equality follows from $p^0(s) = p^\sigma(s)$, for every type profile s and $\sigma \geqslant 0$, which is implied by Assumption $A2'$.

To establish tightness, fix $\gamma < 1$, N and consider the same utility functions and signal profile s^* as in the proof of Proposition 1. Fix $\delta > 0$ and a mechanism that satisfies IR, IC, MC, and is δ-q_efficient. For the function $s_1(x)$ defined as in (3.11), and $s_1^0(x)$ defined the same way but for the efficient allocation $q^0(s)$, similarly as in the proof of Proposition 2 the transfers by player 1 satisfy

$$\begin{aligned}
t_1(s^*) &\leqslant \int_0^{q_1(s^*)} mu_1(x, s_1(x), s^*_{-1})dx \\
&\leqslant \int_0^{q_1(s^*)} mu_1(x, \min\{s^*_1, s^0_1(x(1+\delta))\}, s^*_{-1})dx \\
&\leqslant \int_0^{q^0_1(s^*)/(1+\delta)} mu_1(x, s^0_1(x(1+\delta)), s^*_{-1})dx + \int_{q^0_1(s^*)/(1+\delta)}^{q^0_1(s^*)} mu_1(x, s^*_1, s^*_{-1})dx \\
&= \int_0^{q^0_1(s^*)/(1+\delta)} \left[p^0(s^0_1(x(1+\delta), s^*_{-1}) + x\delta\overline{m}_q\right]dx + \frac{1}{2}\overline{m}_q q^0_1(s^*)^2\left(\frac{\delta}{1+\delta}\right)^2 \\
&= \frac{q^0_1(s^*)^2}{2(1+\delta)}\left(\frac{\delta}{1+\delta}\overline{m}_q - p^{\max}_q\right) + O(\delta^2), \quad p^{\max}_q = \frac{\overline{m}_q}{\underline{m}_o - \phi_N}\left(\frac{\underline{m}_o}{N-1} + \phi_N\right).
\end{aligned}$$

Note that, ignoring the last term $O(\delta^2)$, the last line equals zero precisely when $\delta = \frac{\sigma_N}{\overline{m}_q}$. Given Assumption $A4$, σ_N converges to zero. It follows therefore that for N sufficiently large, if $\delta < \gamma\frac{\sigma_N}{\overline{m}_q}$ then $t_1(s^*) < 0$, and, by an analogous argument, $t_2(s) < 0$. $q^0_i(s^*) = 0$ for $i > 2$ together with IR implies that $\sum_{i>2} t_i(s) \leqslant 0$, and so establish that the mechanism violates BS.

Appendix G: Discrete allocations

In this section, we show how our results can be extended to discrete setting. Specifically, we assume that the feasible allocations $q_i(s)$ are multiples of Δ, for $\Delta \leqslant 1$,

$$q_i(s) \in \{\ldots - 2\Delta, -\Delta, 0, \Delta, 2\Delta, \ldots\} =: \mathbb{Q}_\Delta, \quad \forall i, s.$$

First, the definition of marginal utility must be adjusted to the discrete setting, with $mu_i(q_i, s) = \frac{u_i(q_i+\Delta, s) - u_i(q_i, s)}{\Delta}$, for $q_i \in \mathbb{Q}_\Delta$. Similarly, Assumption $A2$ is now replaced by

$$(A2^d) \qquad mu_i(q_i, s) - mu_i(q_i + \Delta, s) \in [\Delta\underline{m}_q, \Delta\overline{m}_q]. \;\; \forall i, q_i \in \mathbb{Q}_\Delta, s$$

We proceed to define the versions of the σ-VCG mechanisms in discrete setting. Fix a slope σ and a spread 2δ, $\sigma, \delta \geqslant 0$. For any agent i, s and q_i define the tax-adjusted marginal utility $\widetilde{mu}_i(q_i, s)$ as

$$\widetilde{mu}_i(q_i, s) = mu_i(q_i, s) - \sigma q_i - \delta\mathbf{1}_{q_i>0} + \delta\mathbf{1}_{q_i<0}.$$

For the taxes τ that correspond to slope σ and spread 2δ, a τ-VCG allocation $q^\tau(s)$ satisfies

$$\widetilde{mu}_i(q^\tau_i(s) + \Delta, s) \leqslant \widetilde{mu}_j(q^\tau_j(s), s), \;\; \forall i \neq j, \quad \sum_i q^\tau_i(s) = 0. \tag{G.1}$$

The first condition replaces the equality of all the tax-adjusted marginal utilities. It explicitly requires that any additional trade of one unit of size Δ, from agent j to agent $i \neq j$, results in the weak lowering of the sum of their tax-adjusted utilities. In other words, the allocation maximizes the sum of tax-adjusted utilities. As in the main text, when $\sigma = \delta = 0$, the allocation is the efficient Walrasian Equilibrium allocation, for the discrete setting.

The τ-VCG transfers $t^{\tau}(s)$ are defined as

$$t_i^{\tau}(s) = \Delta \sum_{x=\Delta}^{q_i^{\tau}(s)} mu_i\left(x, s_i(x), s_{-i}\right), \quad q_i^{\tau}(s) > 0, \tag{G.2}$$

$$t_i^{\tau}(s) = \Delta \sum_{x=-\Delta}^{q_i^{\tau}(s)} -mu_i\left(x, s_i(x), s_{-i}\right), \quad q_i^{\tau}(s) < 0,$$

where $s_i(x)$ is the pivotal signal for good x,

$$x = q_i^{\tau}\left(s_i(x), s_{-i}\right) \text{ and } x > q_i^{\tau}\left(s_i', s_{-i}\right) \text{ for } s_i' < s_i(x), \quad x > 0,$$

$$x = q_i^{\tau}\left(s_i(x), s_{-i}\right) \text{ and } x < q_i^{\tau}\left(s_i', s_{-i}\right) \text{ for } s_i' > s_i(x), \quad x < 0,$$

$$s_i(x) = \begin{cases} \inf S_i \text{ if } q_i^{\tau}\left(s_i', s_{-i}\right) > x \ \ \forall s_i', \\ \sup S_i \text{ if } q_i^{\tau}\left(s_i', s_{-i}\right) < x \ \ \forall s_i'. \end{cases}$$

The following are the analogues of Propositions 1 and 2, followed by the proofs.[33]

Proposition 6 *Suppose that Assumptions A1, A2^d, A3, and A4 hold. For appropriate $C^d > 0$, there is no family of mechanisms that is robustly asymptotically $C^d(\frac{1}{N} + \phi_N + \Delta)$-efficient.*

More precisely, for every $N > 2$, there are economies of size N that satisfy A1, A2^d, A3, and A4^N, such that for every $\varepsilon < \bar{q}/3$, no mechanism satisfies MC, IR, IC, δ-BS, and ε-Eff, where

$$\delta = N\left[C_1^{dN}\left(\frac{m_o}{N-1} + \phi_N\right) - C_2^d \varepsilon + C_3^{dN}\Delta\right], \quad C_2^d = \overline{q}\overline{m}_q,$$

$$C_1^{dN} = \frac{1}{2}\left(\frac{N-1}{N}\overline{q} - \varepsilon\right)^2 \frac{\overline{m}_q}{\underline{m}_o - \phi_N}, \quad C_3^{dN} = \frac{1}{2}\left(\frac{N-1}{N}\overline{q} - \varepsilon\right)^2 \frac{\overline{m}_q \underline{m}_o}{\underline{m}_o - \phi_N}.$$

Proposition 7 *Suppose that Assumptions A1, A2^d, A3, and A4 hold. For appropriate $D^d > 0$, slopes σ_N^d and spreads $2\delta_N$, the family of the corresponding τ_N-VCG mechanisms is robustly asymptotically $D^d(\frac{1}{N} + \phi_N + \Delta)$-efficient.*

More precisely, for every slope $\sigma \geqslant 0$, spread $2\delta \geqslant 0$, unit size $\Delta \leqslant 1$ and economy of size $N > 2$ that satisfies A1, A2^d, A3, and A4^N, the corresponding τ-VCG mechanism satisfies MC, IR, IC, δ-BS, and ε-efficiency, where

$$\delta = N\left[\left[D_1^{dN}\left(\frac{m_o}{N-1} + \phi_N\right) - D_2^{dN}\sigma\right]_+ + \left[\Delta(D_3^{dN} + D_4^{dN}\sigma) - D_5^d\delta\right]_+\right], \tag{G.3}$$

$$D_1^{dN} = \frac{\overline{q}(\overline{q}+\Delta)}{2}\frac{\overline{m}_q}{\underline{m}_o - \phi_N}, \quad D_2^{dN} = \frac{\overline{q}(\overline{q}+\Delta)}{2}\left(1 - \frac{1}{\underline{m}_o - \phi_N}\left(\frac{m_o}{N-1} + \phi_N\right)\right),$$

$$D_3^{dN} = \frac{\overline{q}\overline{m}_q \underline{m}_o}{\underline{m}_o - \phi_N}, \quad D_4^{dN} = \frac{\overline{q}\underline{m}_o}{\underline{m}_o - \phi_N}, \quad D_5^d = \overline{q}, \quad \varepsilon = D_6^d \sigma + D_7^d \delta.$$

33. In the propositions and the proofs, we use the standard notation, with $[x]_+ = \min\{0, x\}$, $[x]_- = \max\{0, x\}$ and $\lfloor x \rfloor$ the highest integer below x.

Proof of Proposition 6. The proof mirrors that of Proposition 1. Fix N even, and consider the utility functions:

$$u_i(q_i,t_i,s)=((\underline{m}_o-\phi_N)s_i+\phi_N\sum_j s_j)q_i-\frac{\overline{m}_q}{2}q_i^2,\ \ \forall i,q_i,s,$$

and consider the signal profile s^*, such that

$$s_i^*=\widetilde{s}>0,\quad q_i^0(s^*)\in\left[\frac{N-1}{N}\overline{q}-\Delta,\frac{N-1}{N}\overline{q}\right],\quad \forall i\leqslant N/2,$$
$$s_i^*=-\widetilde{s}<0,\quad q_i^0(s^*)=-\left[\frac{N-1}{N}\overline{q}-\Delta,\frac{N-1}{N}\overline{q}\right],\quad \forall i>N/2,$$

together with $mu_i(q_i^0(s^*),s^*)=0$, for all i.

Consider an agent $i\leqslant N/2$ for whom $q_i^0(s^*)>0$. Let us first assume that the values are private, $\phi_N=0$. From the definition of the efficient mechanism and Assumption $A2^d$, agent $i's$ marginal utility at the pivotal type for any of his last $N-1$ units of size Δ, that is, $mu_i(x,s_{-i},s_i(x))$, is at most $\Delta\overline{m}_q$ lower than $mu_i(q_i^\tau(s),s)$. Once $i's$ allocation decreases by $\Delta(N-1)$ (all the other agents bought a unit of size Δ from i already), the marginal utility for the $q_i^\tau(s)-\Delta(N-1)$ unit decreases further by at most $\Delta\left(\overline{m}_q+\sigma\right)$. Consequently, with private values we have

$$mu_i(x,s_{-i},s_i(x))\geqslant mu_i(q_i^\tau(s),s)-\Delta\left(\left\lfloor\frac{q_i^\tau(s)-x}{\Delta(N-1)}\right\rfloor+1\right)\overline{m}_q,\quad x>0.$$

Let us now consider a general, interdependent case, $\phi_N>0$. In this case, as the signal for agent i decreases, so does the marginal utility of other agents. The lowest value for $s_i\left(q_i^\tau(s)\right)$ satisfies, from (G.1),

$$mu_i(q_i^\tau(s),s_{-i},s_i\left(q_i^\tau(s)\right))=mu_j(q_j^\tau(s),s_{-i},s_i\left(q_i^\tau(s)\right))-\Delta\overline{m}_q,$$
$$\left[s_i-s_i\left(q_i^\tau(s)\right)\right]\times\underline{m}_o\leqslant\left[s_i-s_i\left(q_i^\tau(s)\right)\right]\times\phi_N+\Delta\overline{m}_q,$$
$$s_i\left(q_i^\tau(s)\right)\geqslant s_i-\Delta\frac{\overline{m}_q}{\underline{m}_o-\phi_N},$$

and, for any $x\leqslant q_i^\tau(s)$, the pivotal signals satisfy

$$s_i(x)\geqslant s_i-\left(q_i^\tau(s)-x+\Delta+\Delta\left\lfloor\frac{q_i^\tau(s)-x}{\Delta(N-1)}\right\rfloor\right)\frac{\overline{m}_q}{\underline{m}_o-\phi_N},\quad x>0.$$

Following a similar argument in case of a seller, $i>N/2$, given the efficient allocation, we have that for any agent i and any $x\in\mathbb{Q}_\Delta$,

$$mu_i\left(x, s_i^0(x), s_{-i}^*\right) = \tag{G.4}$$

$$= -\Delta\left(\left\lfloor \frac{q_i^0(s)-x}{\Delta(N-1)} \right\rfloor + 1\right)\overline{m}_q - \phi_N\left(z+\Delta+\Delta\left\lfloor \frac{q_i^0(s)-x}{\Delta(N-1)} \right\rfloor\right)\frac{\overline{m}_q}{\underline{m}_o-\phi_N}$$

$$= -\frac{\overline{m}_q}{\underline{m}_o-\phi_N}\left\{\Delta\underline{m}_o+\phi_N\left(q_i^0(s)-x\right)+\Delta\underline{m}_o\times\left\lfloor \frac{q_i^0(s)-x}{\Delta(N-1)} \right\rfloor\right\}, \quad x\leqslant q_i^0(s)$$

$$= \Delta\left(\left\lfloor \frac{x-q_i^0(s)}{\Delta(N-1)} \right\rfloor + 1\right)\overline{m}_q + \phi_N\left(x-q_i^0(s)+\Delta\left\lfloor \frac{x-q_i^0(s)}{\Delta(N-1)} \right\rfloor\right)\frac{\overline{m}_q}{\underline{m}_o-\phi_N}$$

$$= \frac{\overline{m}_q}{\underline{m}_o-\phi_N}\left\{\Delta\left(\underline{m}_o-\phi_N\right)+\phi_N\left(x-q_i^0(s)\right)+\Delta\underline{m}_o\times\left\lfloor \frac{x-q_i^0(s)}{\Delta(N-1)} \right\rfloor\right\}. \quad x>q_i^0(s)$$

Fix $\varepsilon>0$ that is a multiple of Δ[34] and a mechanism that satisfies IR, IC, MC and is ε-efficient. Similarly as in the proof of Proposition 1, the transfers by player 1 satisfy

$$t_1(s^*)\leqslant \Delta\sum_{x=\Delta}^{q_1(s^*)} mu_1(x,s_1(x),s_{-1}^*)\leqslant \Delta\sum_{x=\Delta}^{q_1(s^*)} mu_1(x,\min\{s_1^*,s_1^0(x+\varepsilon)\},s_{-1}^*)$$

$$\leqslant \Delta\sum_{x=\Delta}^{q_1^0(s^*)-\varepsilon} mu_1(x,s_1^0(x+\varepsilon),s_{-1}^*)+\Delta\sum_{x=q_1^0(s^*)-\varepsilon}^{q_1^0(s^*)} mu_1(x,s_1^*,s_{-1}^*)$$

$$\leqslant \varepsilon\overline{m}_q q_1^0(s^*)+\Delta\sum_{x=\Delta}^{q_1^0(s^*)-\varepsilon} mu_1(x+\varepsilon,s_1^0(x+\varepsilon),s_{-1}^*)$$

$$\leqslant \varepsilon\overline{m}_q q_1^0(s^*)-\frac{\overline{m}_q}{\underline{m}_o-\phi_N}\left(\begin{array}{r}\Delta\left(q_1^0(s^*)-\varepsilon\right)\underline{m}_o+\frac{(q_1^0(s^*)-\varepsilon)(q_1^0(s^*)-\varepsilon+\Delta)}{2}\left(\frac{\underline{m}_o}{N-1}+\phi_N\right)\\ -\Delta^2\underline{m}_o\frac{N-2}{2}\times\frac{q_1^0(s^*)-\varepsilon}{\Delta(N-1)}\end{array}\right),$$

where the last inequality follows from (G.4) and the formula

$$\sum_{x=1}^{q}\left\lfloor\frac{x}{N-1}\right\rfloor\geqslant\sum_{x=\Delta}^{q}\frac{x}{N-1}-\frac{N-2}{2}\times\frac{q}{N-1},$$

with the exact equality when x is divisible by $N-1$. Consequently,

$$t_1(s^*)\leqslant C_2^d\times\varepsilon-C_1^{dN}\left(\frac{\underline{m}_o}{N-1}+\phi_N\right)-C_3^{dN}\times\Delta.$$

The same bound holds for the transfers by any buyer $i\leqslant\frac{N}{2}$. Analogous argument shows that the same bound on transfers holds for any seller $i>\frac{N}{2}$. This establishes the proof.

34. Otherwise, in the rest of the proof consider $\varepsilon'>\varepsilon$ that is.

Proof of Proposition 7. Fix a slope $\sigma \geqslant 0$, spread $2\delta \geqslant 0$, profile of utility functions and the corresponding τ-VCG mechanism.

1. The proof that the τ-VCG mechanism satisfies MC, IR and IC is analogous to the corresponding parts of the proof of Proposition 2.

2. τ-VCG mechanism satisfies δ-BS, for δ as in (G.3), and appropriate D_1^d-D_5^d.

Fix s, a τVCG allocation $q^\tau(s)$, agent i, and suppose that $q_i^\tau(s) > 0$. Define

$$\underline{p}^\tau(s) = \min_i \left\{ \widetilde{mu}_i(q_i^\tau(s), s) \right\}, \tag{G.5}$$

$$\overline{p}^\tau(s) = \max_i \left\{ \widetilde{mu}_i(q_i^\tau(s), s) \right\}.$$

Step (i) $\underline{p}^\tau(s) = \overline{p}^\tau(s)$.

Given the same argument as in the proof of Proposition 6 but with $\overline{m}_q$ replaced by $\sigma + \overline{m}_q$ (to account for the increased slope of the modified demand curves), the adjusted marginal utilities at the pivotal types $s_i(x)$, for $x \leqslant q_i^\tau(s)$, satisfy

$$\widetilde{mu}_i(q_i^\tau(s), s) - \widetilde{mu}_i(x, s_i(x), s_{-i}) \leqslant \Delta \left(\left\lfloor \frac{q_i^\tau(s) - x}{\Delta(N-1)} \right\rfloor + 1 \right) (\sigma + \overline{m}_q) \tag{G.6}$$
$$+ \phi_N \left(q_i^\tau(s) - x + \Delta + \Delta \left\lfloor \frac{q_i^\tau(s) - x}{\Delta(N-1)} \right\rfloor \right) \left(\frac{\sigma + \overline{m}_q}{\underline{m}_o - \phi_N} \right),$$

for $x > 0$. The formula implies that the adjusted marginal utility decreases by at most

$$\kappa := \frac{\Delta}{N-1} \left(\underline{m}_o - \phi_N + N \times \phi_N \right) \frac{\sigma + \overline{m}_q}{\underline{m}_o - \phi_N} = \Delta \left(\frac{\underline{m}_o}{N-1} + \phi_N \right) \frac{\sigma + \overline{m}_q}{\underline{m}_o - \phi_N}, \tag{G.7}$$

on average for every unit of size Δ purchased (compare with line (B.5) in the proof of Proposition 2). Moreover, the difference $\widetilde{mu}_i(q_i^\tau(s), s) - \widetilde{mu}_i(x, s_{-i}, s_i(x))$ between the adjusted marginal utilities is always smaller than $\Delta(\sigma + \overline{m}_q) + \kappa \left(\frac{q_i^\tau(s) - x}{\Delta} + 1 \right)$. Just as in (B.6) in the proof of Proposition 2, this implies that the transfers satisfy

$$t_i^\tau(s) = \Delta \sum_{x=\Delta}^{q_i^\tau(s)} mu_i(x, s_i(x), s_{-i}) = \Delta \sum_{x=\Delta}^{q_i^\tau(s)} \left\{ \widetilde{mu}_i(x, s_i(x), s_{-i}) + \sigma x + \delta \right\} \tag{G.8}$$
$$\geqslant q_i^\tau(s) \underline{p}^\tau(s) + \Delta \sum_{x=\Delta}^{q_i^\tau(s)} \left\{ -\kappa \left(\frac{q_i^\tau(s) - x}{\Delta} + 1 \right) - \Delta(\sigma + \overline{m}_q) + \sigma x + \delta \right\}$$
$$\geqslant q_i^\tau(s) \underline{p}^\tau(s) + q_i^\tau(s) \left[\delta - \Delta(\sigma + \overline{m}_q) \right] + \Delta \left(\Delta + 2\Delta + \ldots + q_i^\tau(s) \right) \times \left[\sigma - \frac{\kappa}{\Delta} \right]$$
$$\geqslant q_i^\tau(s) \underline{p}^\tau(s) + \overline{q} \left[\delta - \Delta(\sigma + \overline{m}_q) \right]_- + \frac{\overline{q}(\overline{q} + \Delta)}{2} \left[\sigma - \left(\frac{\underline{m}_o}{N-1} + \phi_N \right) \frac{\sigma + \overline{m}_q}{\underline{m}_o - \phi_N} \right]_- .$$

Step (ii) Let us now consider the general case, when at the τ-VCG allocation the adjusted marginal utilities are not equal, $\underline{p}^\tau(s) = \overline{p}^\tau(s)$. From the definition of the τ-VCG allocation, the distance between $\underline{p}^\tau(s)$ and $\overline{p}^\tau(s)$ is at most $\Delta(\sigma + \overline{m}_q)$.

In this case, we need the following adjustments. First, in the private value case, the bound changes to

$$\widetilde{mu}_i(x,s_{-i},s_i(x))\geqslant \underline{p}^\tau(s)-\Delta\left(\left\lfloor\frac{q_i^\tau(s)-x}{\Delta(N-1)}\right\rfloor+1\right)\left(\sigma+\overline{m}_q\right),\quad x>0.$$

Second, given that the difference between $\widetilde{mu}_i(q_i^\tau(s),s)$ and $\underline{p}^\tau(s)$ is at most $\sigma+\overline{m}_q$, similarly as in the proofs of Proposition 6 and step (i), for $x>0$

$$s_i\left(q_i^\tau(s)\right)\geqslant s_i-2\Delta\frac{\sigma+\overline{m}_q}{\underline{m}_o-\phi_N},$$

$$s_i(x)\geqslant s_i-\left(q_i^\tau(s)-x+2\Delta+\Delta\left\lfloor\frac{q_i^\tau(s)-x}{\Delta(N-1)}\right\rfloor\right)\left(\frac{\sigma+\overline{m}_q}{\underline{m}_o-\phi_N}\right),$$

$$\underline{p}^\tau(s)-\widetilde{mu}_i(x,s_{-i},s_i(x))\leqslant\Delta\left(\left\lfloor\frac{q_i^\tau(s)-x}{\Delta(N-1)}\right\rfloor+1\right)\left(\sigma+\overline{m}_q\right)$$
$$+\phi_N\left(q_i^\tau(s)-x+2\Delta+\Delta\left\lfloor\frac{q_i^\tau(s)-x}{\Delta(N-1)}\right\rfloor\right)\left(\frac{\sigma+\overline{m}_q}{\underline{m}_o-\phi_N}\right).$$

Altogether, we have

$$\underline{p}^\tau(s)-\widetilde{mu}_i(x,s_{-i},s_i(x))\leqslant\Delta\left(\sigma+\overline{m}_q+\phi_N\left(\frac{\sigma+\overline{m}_q}{\underline{m}_o-\phi_N}\right)\right)+\kappa\left(\frac{q_i^\tau(s)-x}{\Delta}+1\right)$$
$$=\Delta\frac{\left(\sigma+\overline{m}_q\right)\underline{m}_o}{\underline{m}_o-\phi_N}+\kappa\left(\frac{q_i^\tau(s)-x}{\Delta}+1\right),$$

for κ as in (G.7). Consequently, similarly as in the proof of Proposition 2, we have

$$t_i^\tau(s)\geqslant q_i^\tau(s)\underline{p}^\tau(s)+\left[\delta-\Delta\frac{\left(\sigma+\overline{m}_q\right)\underline{m}_o}{\underline{m}_o-\phi_N}\right]q_i^\tau(s)+\left[\sigma-\frac{\kappa}{\Delta}\right]\frac{q_i^\tau(s)\left(q_i^\tau(s)+\Delta\right)}{2} \tag{G.9}$$
$$\geqslant q_i^\tau(s)\underline{p}^\tau(s)+\left[D_2^d\times\sigma-D_1^d\left(\frac{\underline{m}_o}{N-1}+\phi_N\right)\right]_-+\left[D_5^d\times\delta-\Delta(D_3^d+D_4^d\times\sigma)\right]_-.$$

Since the same inequality holds for every seller i, with $q_i^\sigma(s)<0$, adding up the inequalities establishes the proof of part 2.

3. τ-VCG mechanism satisfies ε-Eff, for ε as in (G.3), and appropriate D_6^d, D_7^d.

Fix an economy, the τ-VCG mechanism, for a given slope σ and spread 2δ, as well as an s. First, similarly as in the proof of Proposition 2 we have

$$\max_i mu_i(q_i^0(s),s)\leqslant\max_i mu_i(q_i^\tau(s),s)+\delta,$$
$$\min_i mu_i(q_i^0(s),s)\geqslant\min_i mu_i(q_i^\tau(s),s)-\delta.$$

This in turn implies that

$$\max_i \left|q_i^\tau(s)-q_i^0(s)\right| \leqslant \frac{1}{\underline{m}_q}\max_i \left|mu_i(q_i^\tau(s),s)-mu_i(q_i^0(s),s)\right|$$
$$\leqslant \frac{1}{\underline{m}_q}\left(\max_i mu_i(q_i^\tau(s),s)-\min_i mu_i(q_i^\tau(s),s)+\delta\right)$$
$$\leqslant \frac{1}{\underline{m}_q}\left(2\sigma \max_i \left|q_i^\tau(s)\right|+3\delta\right) \leqslant \frac{1}{\underline{m}_q}(2\sigma\overline{q}+3\delta),$$

which implies the second line in (G.3) with

$$D_6^d=\frac{2\overline{q}}{\underline{m}_q}, \quad D_7^d=\frac{3}{\underline{m}_q}.$$

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